# Stability and Control of Flight Vehicle

Uy-Loi Ly
Department of Aeronautics and Astronautics, Box 352400
University of Washington
Seattle, WA 98195

September 29, 1997

©Copyright 1997, by Uy-Loi Ly. All rights reserved.

No parts of this book may be photocopied or reproduced in any form without the written permission.

#### **Contents**

| Gl | Glossary viii |   |    |
|----|---------------|---|----|
| 1  | Intra         | oduction  | 1  |
| •  | 1.1           | Lift Equation   | 2  |
|    | 1.2           | Drag Equation   | 3  |
|    | 1.3           | Pitching Moment Equation  | 4  |
|    | 1.5           | Thening Woment Equation   | 7  |
| 2  | Line          | ear Algebra and Matrices  | 5  |
|    | 2.1           | Operations  | 7  |
|    | 2.2           | Matrix Functions  | 8  |
|    | 2.3           | Linear Ordinary Differential Equations                                | 11 |
|    | 2.4           | Laplace Transform   | 13 |
|    | 2.5           | Stability   | 16 |
|    | 2.6           | Example   | 17 |
|    |               | 2.6.1 Laplace method  | 17 |
|    |               | 2.6.2 Time-domain method  | 18 |
|    |               | 2.6.3 Numerical integration method (via MATLAB)                       | 20 |
| 3  | Prin          | nciples of Static and Dynamic Stability                               | 23 |
| 4  | Stati         | ic Longitudinal Stability   | 27 |
| •  | 4.1           | Notations and Sign Conventions  | 27 |
|    | 4.2           | Stick-Fixed Stability   | 27 |
|    | 4.3           | Stick-Free Stability  | 34 |
|    | 4.4           | Other Influences on the Longitudinal Stability                        | 39 |
|    | 7.7           | 4.4.1 Influence of Wing Flaps   | 39 |
|    |               | 4.4.2 Influence of the Propulsive System                              | 40 |
|    |               | 4.4.3 Influence of Fuselage and Nacelles                              | 42 |
|    |               | 4.4.4 Effect of Airplane Flexibility                                  | 42 |
|    |               | 4.4.5 Influence of Ground Effect                                      | 43 |
|    |               | 4.4.5 Influence of Ground Effect                                      | 73 |
| 5  | Stati         | ic Longitudinal Control   | 45 |
|    | 5.1           | Longitudinal Trim Conditions with Elevator Control                    | 45 |
|    |               | 5.1.1 Determination of Elevator Angle for a New Trim Angle of Attack  | 48 |
|    |               | 5.1.2 Longitudinal Control Position as a Function of Lift Coefficient | 48 |

|    | <b>7</b> 0 |  |
|----|------------|--|
|    | 5.2        | Control Stick Forces   |
|    |            | 5.2.1 Stick Force for a Stabilator                                 |
|    | <b>5</b> 0 | 5.2.2 Stick Force for a Stabilizer-Elevator Configuration          |
|    | 5.3        | Steady Maneuver  |
|    |            | 5.3.1 Horizontal Stabilizer-Elevator Configuration: Elevator per g |
|    |            | 5.3.2 Horizontal Stabilator Configuration: Elevator per g          |
|    |            | 5.3.3 Stabilizer-Elevator Configuration: Stick Force per g         |
|    |            | 5.3.4 Stabilator Configuration: Stick Force per g                  |
| 6  | Late       | eral Static Stability and Control                                  |
|    | 6.1        | Yawing and Rolling Moment Equations                                |
|    |            | 6.1.1 Contributions to the Yawing Moment                           |
|    |            | 6.1.2 Contributions to the Rolling Moment                          |
|    | 6.2        | Directional Stability (Weathercock Stability)                      |
|    | 6.3        | Directional Control  |
|    | 6.4        | Roll Stability   |
|    | 6.5        | Roll Control   |
| 7  | Davi       | iew of Rigid Body Dynamics 75                                      |
| ,  | 7.1        | Few of Rigid Body Dynamics 75  Force Equations                     |
|    | 7.1        | Moment Equations   |
|    | 7.3        | Euler's Angles   |
|    | 1.5        | Luiei s'Aligies  |
| 8  |            | earized Equations of Motion 85                                     |
|    | 8.1        | Linearized Linear Acceleration Equations                           |
|    | 8.2        | Linearized Angular Acceleration Equations                          |
|    | 8.3        | Linearized Euler's Angle Equations                                 |
|    | 8.4        | Forces and Moments in terms of their Coefficient Derivatives       |
|    |            | 8.4.1 Lift Force <i>L</i>  |
|    |            | 8.4.2 Drag Force $D$   |
|    |            | 8.4.3 Side-Force <i>Y</i>  |
|    |            | 8.4.4 Thrust Force <i>T</i>  |
|    |            | 8.4.5 Pitching Moment $M$  |
|    |            | 8.4.6 Yawing Moment $N$  |
|    |            | 8.4.7 Rolling Moment $L$   |
| 9  | Line       | earized Longitudinal Equations of Motion 97                        |
|    | 9.1        | Phugoid-Mode Approximation   |
|    | 9.2        | Short-Period Approximation   |
| 10 | Line       | earized Lateral Equations of Motion 109                            |
|    |            | •  |
| 11 |            | ht Vehicle Models  |
|    |            | Generic F-15 Model Data (Supersonic)                               |
|    | 11/        | A TENERIC DE LA IMPONEL DATA LA MODERSONICI                        |

### **List of Figures**

| 1.1  | Motion in the Longitudinal Axis  | 2   |
|------|--|-----|
| 3.1  | Three Possible Cases of Static Stability   | 24  |
| 3.2  | Three Possible Cases of Dynamic Stability  | 25  |
| 4.1  | Moments about the Center of Gravity of the Airplane  | 28  |
| 4.2  | Definition of Aircraft Variables in Flight Mechanics   | 28  |
| 4.3  | Forces and Moments Applied to a Wing-Tail Configuration  | 28  |
| 4.4  | Moment Coefficient $C_{M_{cg}}$ versus $\alpha$  | 30  |
| 4.5  | Calculation of Wing Aerodynamic Center   | 33  |
| 4.6  | Horizontal Tail Configurations   | 34  |
| 4.7  | Forces on a Propeller  | 40  |
| 4.8  | Propeller Normal Force Coefficient $C_{N_{p\alpha}} = \frac{\partial C_{N_{blade}}}{\partial \alpha} f(T)$ | 41  |
| 4.9  | $K_f$ as a Function of the Position of the Wing $c/4$ Root Chord   | 42  |
| 5.1  | How to Change Airplane Trim Angle of Attack  | 46  |
| 5.2  | Tail Lift Coefficient vs Tail Angle of Attack  | 46  |
| 5.3  | Tail Lift Coefficient vs Elevator Deflection   | 47  |
| 5.4  | Determination of Stick-Fixed Neutral Point from Flight Test  | 50  |
| 5.5  | Longitudinal Control Stick to Stabilator   | 50  |
| 5.6  | Stick Force versus Velocity Curve  | 53  |
| 6.1  | Definition of the Lateral Directional Motion of an Airplane  | 62  |
| 6.2  | Effect of Sweepback on Total Lift and Rolling Moment to Sideslip   | 68  |
| 6.3  | Effect of Wing Placement on the Rolling Moment to Sideslip   | 69  |
| 6.4  | Airplane with a Positive Sideslip  | 73  |
| 7.1  | Motion of a Rigid Body   | 76  |
| 7.2  | Euler's Angle Definition   | 80  |
| 8.1  | Definition of Angle of Attack $\alpha$ and Sideslip $\beta$  | 85  |
| 8.2  | X and $Z$ -Force Components in terms of $L$ , $D$ and $T$  | 87  |
| 9.1  | Longitudinal Aircraft Responses to a 1-deg Elevator Impulsive Input  | 104 |
| 9.2  | Short-Period Approximation Model to a 1-deg Elevator Impulse Input   | 107 |
| 10.1 | Lateral Responses to a 1-deg Aileron Impulse Input   | 115 |

| LIST OF FIGURES  | v |
|--|---|
|  |   |
| 10.2 Lateral Responses to a 1-deg Rudder Impulse Input |   |

#### **List of Tables**

| 2.1        | Laplace Transforms of Some Common Functions | 13 |
|------------|---|----|
| <b>4.1</b> | Laplace Transforms of Some Common Functions | 13 |

viii Glossary

#### Glossary

- a Lift curve slope (1/rad)
- a Speed of sound (ft/sec)
- b Wing span (ft)
- c Wing chord (ft)
- $\bar{c}$  Mean aerodynamic chord (ft)
- D Drag force (lbs)
- **F** Total force (lbs)
- $F_x$  Force component along the x-axis (lbs)
- $F_v$  Force component along the y-axis (lbs)
- $F_z$  Force component along the z-axis (lbs)
- g Gravitational acceleration (ft/sec<sup>2</sup>)
- *H* Hinge moment (ft-lbs)
- I Identity matrix
- I Inertia matrix (slugs-ft<sup>2</sup>)
- $I_{xx}$  Moment of inertia about the x-axis (slugs-ft<sup>2</sup>)
- $I_{yy}$  Moment of inertia about the y-axis (slugs-ft<sup>2</sup>)
- $I_{zz}$  Moment of inertia about the z-axis (slugs-ft<sup>2</sup>)
- L Lift force (lbs)
- L Rolling moment (ft-lbs)
- m Vehicle mass (slugs)
- M Mach number (dimensionless)
- M Total moment (ft-lbs)
- $M_x$  Moment about the vehicle x-axis (ft-lbs)
- $M_{\nu}$  Moment about the vehicle y-axis (ft-lbs)
- $M_z$  Moment about the vehicle z-axis (ft-lbs)
- M Pitching moment (ft-lbs)
- N Yawing moment (ft-lbs)
- p Roll rate (rad/sec)
- q Pitch rate (rad/sec)
- q Dynamic pressure (psi)
- r Yaw rate (rad/sec)
- S Surface area ( $ft^2$ )
- t Time (sec)
- T Thrust force (lbs)
- u Velocity component along the x-axis (ft/sec)
- v Velocity component along the y-axis (ft/sec)
- V Aircraft velocity vector (ft/sec)
- V Velocity (ft/sec)
- $V_H$  Horizontal tail volume (dimensionless)

Glossary ix

|                      | <b>T</b> 7 1 2 4 1 2 (6/ )                                      |
|----------------------|---|
| w                    | Velocity component along the <i>z</i> -axis (ft/sec)            |
| W                    | Vehicle weight (lbs)  |
| X                    | Force along the $x$ -axis (lbs)                                 |
| Y                    | Side force or force along the <i>y</i> -axis (lbs)              |
| Z                    | Force along the z-axis (lbs)                                    |
| <b>Greek Symbols</b> |   |
| $\alpha$             | Angle of attack (rad)   |
| $oldsymbol{eta}$     | Sideslip angle (rad)  |
| δ                    | Surface deflection (rad)  |
| $\epsilon$           | Downwash angle (rad)  |
| γ                    | Flight path angle (rad)   |
| λ                    | Taper ratio (dimensionless)                                     |
| ho                   | Air density (slugs/ft <sup>3</sup> )                            |
| $ar{\sigma}$         | Maximum singular value  |
| heta                 | Pitch angle (rad)   |
| <b>Subscripts</b>    |   |
| cg                   | Center of gravity   |
| ac                   | Aerodynamic center  |
| a                    | Aileron   |
| e                    | Elevator  |
| r                    | Rudder  |
| w                    | Wing  |
| t                    | Tail  |
| <b>Operators</b>     |   |
| $E\left[ st  ight]$  | Expected value  |
| $\dot{x}$            | Time derivative of the variable <i>x</i>                        |
| $x_i$                | $i^{th}$ element of the vector $x$                              |
| $A_{ij}$             | Element of the A matrix in the $i^{th}$ row and $j^{th}$ column |
|                      |   |

x Glossary

## **Chapter 1**

### Introduction

The objective of this course is to develop fundamental understanding on the subject of stability, control and flight mechanics. Familiarities with the basic components in aerodynamics of wing and airfoil section are expected; namely definition of lift, drag and moment of wing section, and physical parameters that govern these aerodynamic forces and moments suchas freestream velocity, density, Mach number, Reynold number, shape of the airfoil (camber, thickness, aerodynamic center), wing configuration (wing span, reference area, mean aerodynamic chord, taper ratio, sweep angle), angle of attack, dynamic pressure, etc... It is not the intent of this course to provide all these relevant background materials, although we will define the relevant ones as we encounter them in our problem formulation.

Starting from known forces and moments generated on a given wing, fuselage and tail configuration, we will develop airplane static and dynamic model to study its behavior under different flight regimes. Mass properties, wing, fuselage and tail configurations of the airplane are therefore assumed known and given a-priori. Concepts of static stability and dynamic stability are introduced in this course. General equations of motion for a rigid-body airplane are derived. Basic motions of the aircraft separated into longitudinal and lateral modes are discussed in details. Effects of aerodynamic stability derivatives upon the behaviour of the perturbed equations of motions are studied. Flying qualities of the uncontrolled airplane can subsequently be assessed. Analysis of the airplane dynamic responses to initial changes in its basic motion variables (e.g. angle of attack, pitch attitude, roll angle, sideslip, etc···), tocontrol inputs and external gust inputs iscovered using Laplace transform techniques and time simulation. It is expected that students are familiar somewhat with the use of the MATLAB software for analysis.

There is an extensive number of references that cover the subject of aircraft stability, control and flight mechanics. Listed below are some that provide good reference materials:

- 1. John D. Anderson, Jr., Introduction to Flight, Third Edition, McGraw-Hill Book Company, 1989.
- 2. Arthur W. Babister, Aircraft Dynamic Stability and Response, First Edition, Pergamon Press, 1980.
- 3. John H. Blakelock, *Automatic Control of Aircraft and Missiles*, Second Edition, John Wiley & Sons, Inc., 1991.
- 4. Bernard Etkin, *Dynamics of Flight: Stability and Control*, Second Edition, John Wiley & Sons, Inc., 1982.
- 5. Barnes W. McCormick, *Aerodynamics, Aeronautics, and Flight Mechanics*, John Wiley & Sons, Inc., 1979.

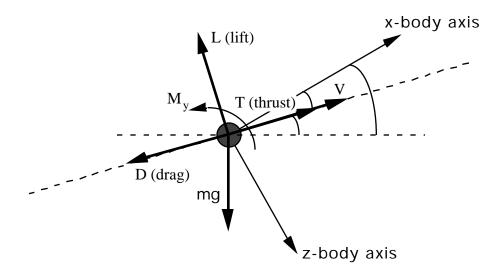


Figure 1.1: Motion in the Longitudinal Axis

- Courtland, D. Perkins and Robert E. Hage, Airplane Performance, Stability and Control, John Wiley & Sons, Inc., 1949.
- 7. Jan Roskam, *Airplane Flight Dynamics and Automatic Controls*, Roskam Aviation and Engineering Corporation, 1979.
- 8. Edward Seckel, Stability and Control of Airplanes and Helicopters, Academic Press, 1964.
- 9. David R. Hill and Clever B. Moler, *Experiments in Computational Matrix Algebra*, Random House, First Edition, 1988.

As an introduction, let's first examine the development of the following three basic equations corresponding to motion in the vertical plane (i.e., longitudinal set). They correspond to the lift, drag and moment equations respectively.

### 1.1 Lift Equation

Let's consider the point mass system shown in Figure 1.1. Here we idealize the airplane as a lumped system with mass m and moment of inertia about the y-axis as  $I_{yy}$ . Note that the flight pathisal waystangential to the velocity vector  $\mathbf{V}$ . The aerodynamic forces applied to the center of mass of the vehicle can be decomposed into the lift and drag components. The lift force is by definition perpendicular to the velocity vector  $\mathbf{V}$  while the drag force is parallel to the velocity vector  $\mathbf{V}$  and is pointed in the opposite direction. Gravity force  $m\mathbf{g}$  and the engine thrust  $\mathbf{T}$  constitute the remaining forces exerted on the vehicle. The pitching moment  $\mathbf{M}_y$  about the airplane center of gravity (cg) is mainly due to aerodynamic and propulsion forces and has no contribution from gravity.

The equation of motion along the z-body axis is given by

$$m(\dot{w} - qu) = \Sigma F_z \tag{1.1}$$

where u is the component of velocity along the x-body axis of the vehicle, w is the component along the z-body axis and q is the pitch angular velocity about the y-body axis. From Figure 1.1, we have

$$\sum F_z = F_{z(aerodynamics)} + F_{z(propulsion)} + F_{z(gravity)}$$
(1.2)

where

$$F_{z(aerodynamics)} + F_{z(propulsion)} = -Lcos \alpha + (T - D)sin \alpha$$

$$\cong -L + (T - D)\alpha \quad \text{(for small } \alpha\text{)}$$
(1.3)

$$F_{z(gra\ vity)} = mgcos\theta$$

$$\cong W \quad \text{(for small }\theta\text{)}$$
(1.4)

where W is the weight of the vehicle. Let's rewrite the velocity component w as  $w = V \sin \alpha$  and thus,

$$\dot{w} = \dot{V}\sin\alpha + V\dot{\alpha}\cos\alpha 
\cong \dot{V}\alpha + V\dot{\alpha} \quad \text{(for small }\alpha\text{)}$$
(1.5)

Generally, we notice that the product  $\dot{V}\alpha$  is much smaller than  $V\dot{\alpha}$ . Hence, equation (1.5) is simplified to

$$\dot{w} = V\dot{\alpha} \tag{1.6}$$

Combining equations (1.1), (1.2), (1.3), (1.4) and (1.6), we obtain the following equation in the z-body direction,

$$m(V\dot{\alpha} - \dot{\theta}V) = -L + (T - D)\alpha + W \tag{1.7}$$

since  $q = \dot{\theta}$  and  $u = V\cos\alpha \cong V$ . Usually, the terms T - D and  $\alpha$  are small and hence we can drop the product  $(T - D)\alpha$  in the above equation. Thus,

$$mV(\dot{\alpha} - \dot{\theta}) = W - L \tag{1.8}$$

Note that the flight path angle is defined as  $\gamma = \theta - \alpha$ , equation (1.8) can be rewritten as

$$mV\dot{\gamma} = L - W \tag{1.9}$$

Thus, change in flight path occurs when  $L - W \neq 0$  and the corresponding flight trajectory would be curved. For a constant flight path angle (i.e.  $\gamma = \gamma_o$  =constant), we must have  $\dot{\gamma} = 0$  and L - W = 0.

### 1.2 Drag Equation

Again we refer to Figure 1.1, the equation of motion in the x-body direction is as follows,

$$m(\dot{u} + qw) = \Sigma F_x \tag{1.10}$$

since we are limited to motion in the vertical plane only. The force components in the x-body direction are only consisted of  $\Sigma F_x = F_{x(aerodynamics)} + F_{x(propulsion)} + F_{x(gravity)}$ . Each of these components can again be written in terms of the lift L, drag D, thrust T and gravity W forces according to Figure 1.1. Namely,

$$F_{x(aerodynamics)} + F_{x(propulsion)} = Lsin \alpha + (T - D)cos\alpha$$
  
 $\cong L\alpha + (T - D)$  (for small  $\alpha$ ) (1.11)

and

$$F_{x(gra\ vity)} = -mgsin\ \theta \approx -W\theta \quad \text{(for small }\theta\text{)}$$
 (1.12)

Furthermore, the velocity component u can be rewritten as  $u = V\cos\alpha$ . Hence, its time derivative becomes

$$\dot{u} = \dot{V}\cos\alpha - V\dot{\alpha}\sin\alpha 
= \dot{V} - V\dot{\alpha}\alpha \quad \text{(for small }\alpha\text{)}$$
(1.13)

Substituting equations (1.11), (1.12) and (1.13) into equation (1.10), we obtain the following

$$m\dot{V} + m\alpha(V\dot{\theta} - V\dot{\alpha}) = T - D + (L - W)\alpha - W(\theta - \alpha) \tag{1.14}$$

or,

$$m\dot{V} + W\gamma = T - D + \alpha(L - W - mV\dot{\gamma}) \tag{1.15}$$

Using equation (1.9), equation (1.15) is simplified to

$$m\dot{V} + W\gamma = T - D \tag{1.16}$$

Thus from the above equation with excess thrust, i.e. (T - D) > 0, one can have different flight trajectories:

- Positive flight path angle  $\gamma > 0$  with  $\dot{V} = 0$ . This results in a steady (nonaccelerated) climb.
- Positive acceleration  $\dot{V} > 0$  with  $\gamma = 0$ . This corresponds to an accelerated straight and level flight.
- Positive acceleration  $\dot{V} > 0$  and  $\gamma > 0$ . The vehicle speed increases while climbing.

### 1.3 Pitching Moment Equation

Finally, we derive the pitching moment equation for the vehicle shown in Figure 1.1,

$$I_{yy}\ddot{\theta} = M_{y(aerodynamics)} + M_{y(propulsion)}$$
 (1.17)

Notice that by definition, gravity would have no moment contribution to the pitching moment equation when it is taken about the vehicle center of gravity. Detailed description of the moments produced by aerodynamic and propulsive forces will be given later when we examine issues related to longitudinal static stability. It suffices to say that static longitudinal stability is predominantly governed by the behaviour of the *pitching moment* as the vehicle is perturbed from its equilibrium state.

## **Chapter 2**

## **Linear Algebra and Matrices**

We deal with 3 classes of numbers: *scalars*, single numbers without association; *vectors*, one dimensional groupings of scalars (one column, several rows, or one row, several columns); and finally, *matrices*, which for us will be 2-dimensional (rows and columns).

A vector can be either a row vector such as

$$\vec{s} = [s_1 \ s_2 \cdots s_n],$$

or a column vector, such as

$$\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}.$$

Column vectors are vastly more common. Implied with every vector is a basis (often a physical basis) to which each component refers. For instance, the position vector  $\vec{r}$  given by

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

is represented with respect to a cartesian basis  $[\hat{x}, \hat{y}, \hat{z}]$ . Usually the shorthand [x, y, z] is used.

MATLAB will use both row and column vectors. However column vectors are more often used in its functions. Note that in the example below, the first entry in boldface is what the user types, and the second corresponds to MATLAB's response.

- **A** = [1. 2. 3. 4.] **A** = 1 2 3 4
- A = [1. 2. 3. 4.]'
  - A =
  - 1
  - 2
  - 3
  - 4

• A = [1;2;3;4]

A =

1

2

3

• A = [1]

2

3

**4**]

A=

1 2

3

4

A matrix can be thought of as a row of column vectors, or a column of row vectors,

$$A = \begin{bmatrix} \hat{a}_1 \ \hat{a}_2 \ \cdots \ \hat{a}_m \end{bmatrix} = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

One can think of matrices as having a basis in the form of *dyadic products* of basis vectors, though that will be beyond the scope of this course.

Entry of a matrix in MATLAB is fairly straightforward. It follows along the lines of a vector, but remember that the entries are processed in row fashion:

•  $A = [1 \ 2 \ 3 \ 4; 5 \ 6 \ 7 \ 8; 9 \ 10 \ 11 \ 12]$ 

A=

1234

5678

9 10 11 12

• A = [1234]

5678

9 10 11 12]

A=

1234

5678

9 10 11 12

• B = [A [0 1;2 3;4 5]; 6 5 4 3 2 1; 3 4 5 6 7 8; 1 3 5 7 9 11]

B =

123401

567823

2.1. OPERATIONS 7

9 10 11 12 4 5 6 5 4 3 2 1 3 4 5 6 7 8 1 3 5 7 9 11

MATLAB also has facilities for creating simple matrices such as a matrix of zeros or the identity matrix. For example, the matrix function zeros(n,m) will create a zero matrix of dimension n by m, and eye(n) will create an identity matrix of dimension n.

### 2.1 Operations

Addition and multiplication not only need to be defined for within a certain class, but between classes. For example, multiplication of vectors and matrices by a scalar,

$$\alpha * \vec{v} = \begin{bmatrix} \alpha s_1 \\ \alpha s_2 \\ \vdots \\ \alpha s_n \end{bmatrix}; \quad \alpha * M = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1m} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{n1} & \alpha a_{n2} & \cdots & \alpha a_{nm} \end{bmatrix}.$$

In MATLAB, however, you can add a scalar to every element in a vector or matrix without any special notation.

Adding two vectors occurs on an element by element level. Implied in all of this is that the basis of the two vectors is the same:

$$\vec{v} + \vec{u} = [v_1 \quad v_2 \quad \cdots \quad v_n] + [u_1 \quad u_2 \quad \cdots \quad u_n]$$
  
=  $[v_1 + u_1 \quad v_2 + u_2 \quad \cdots \quad v_n + u_n]$ 

Multiplication of two vectors is mostly defined in terms of the *dot product*. While much mathematical theory has been expounded on inner product spaces and such, the only item we need know here is the inner product of two vectors expressed in a Cartesian coordinate frame,

$$\vec{v} \cdot \vec{u} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} * \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$
$$= v_1 u_1 + v_2 u_2 + \cdots + v_n u_n$$

Multiplying a vector by a matrix is equivalent to transforming the vector. In components, the product of a matrix and a vector is given by

$$A * \vec{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} * \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1m}v_m \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2m}v_m \\ \vdots \\ a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nm}v_m \end{bmatrix}$$

Of course, the number of columns of A must match the number of rows of  $\vec{v}$ .

Adding two matrices of the same dimensions would occur on an element by element basis,

$$A + B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1m} + b_{1m} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2m} + b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nm} + b_{nm} \end{bmatrix}$$

Multiplying two matrices can be thought of as a series of transformations on the column vectors of the multiplicand. The number of columns of the left matrix must be equal to the number of rows of the right matrix (*left* and *right* are significant since multiplication is not commutative for matrices in general).

$$A * B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} * \begin{bmatrix} \hat{b}_{1} & \hat{b}_{2} & \cdots & \hat{b}_{p} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1m}b_{m2} & \cdots \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2m}b_{m1} & a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2m}b_{m2} & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}b_{11} + a_{n2}b_{21} + \cdots + a_{nm}b_{m1} & a_{n1}b_{12} + a_{n2}b_{22} + \cdots + a_{nm}b_{m2} & \vdots \end{bmatrix}$$

The standard symbols: +, -, \*, and/will handle all legal operations between scalars, vectors, and matrices in MATLAB without any further special notation.

#### 2.2 Matrix Functions

The convenience of matrix notation is in the representation of a group of linear equations. For example, the following set of equations,

$$\begin{cases} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = b_1 \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & = b_2 \\ & & \vdots & & \vdots \\ a_{n1}x_1 & +a_{n2}x_2 & +\cdots & +a_{nn}x_n & = b_n \end{cases}$$

can be represented compactly by the relation

$$A\vec{x} = \vec{b}$$

Note that we have the number of knowns  $\vec{b}$  equal to the number of unknowns  $\vec{x}$  here. How does one solve this? The most simplistic (and computationally efficient) method is to apply a succession of transformations

to the above system to eliminate values of A below the diagonal. Suppose that  $a_{11}$  is nonzero then one can multiply the first equation by  $a_{21}/a_{11}$  and subtracts it from the second equation. The first term in the second equation would be eliminated,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ 0 + (a_{22} - a_{12}a_{21}/a_{11})x_2 + \cdots + (a_{2n} - a_{1n}a_{21}/a_{11})x_n = b_2 - b_1a_{21}/a_{11} \\ \vdots + \vdots + \vdots + \vdots + \vdots = \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

Suppose that the procedure were repeated for all the other rows, resulting in the removal of the coefficient of  $x_1$  in these equations. The same procedure is now applied to all coefficients of  $x_2$  for all rows below the second row. What one would eventually have is the *upper triangular* system. (In general, the  $\tilde{a}_{ij}$ 's and  $\tilde{b}_j$ 's are NOT the same as the original matrix entries  $a_{ij}$  and  $b_j$ , respectively).

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ 0 + \tilde{a}_{22}x_2 + \cdots + \tilde{a}_{2n}x_n &= b_2 - b_1a_{21}/a_{11} \\ 0 + 0 + \cdots + \cdots &= \cdots \\ \vdots + \vdots + \vdots + \vdots + \vdots &= \vdots \\ 0 + 0 + \cdots + \tilde{a}_{nn}x_n &= \tilde{b}_n \end{cases}$$

All values of  $\tilde{a}_{ij}$  below the diagonal are zero. This matrixalso has the interesting propertythat the product of the diagonal terms is equal to the *determinant*. This substantiates the argument that it costs about as much to solve a linear system as it does to solve for a determinant. Note that one can now solve  $x_n = \tilde{b}_n/\tilde{a}_{nn}$ . Once you have  $x_n$ , you can substitute it into the next equation up and solve for  $x_{n-1}$ . This continues until one gets to  $x_1$  (or until some diagonal term  $\tilde{a}_{kk}$  is zero). Note that if a diagonal term is zero, then the determinant is also zero and the system matrix A is called singular.

The above method is often called the method of Gaussian elimination with back substitution. MATLAB implements a method very much similar to the above for solving a system of linear equations. You can, however, obtain an answer from MATLAB with very little effort by just typing the command

$$\mathbf{x} = \mathbf{A} \setminus \mathbf{b}$$

In a similar vein to whatis noted above, the determinant is computed with the following command syntax **det(A)**.

The inverse can also be computed in a method similar to that above. If one solves for a succession of vectors  $\vec{x}_i$  (i = 1, n), each one with

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \quad \vec{b}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}; \quad \cdots \quad \vec{b}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

then the matrix composing of the column vectors  $[\vec{x_1}, \cdots \vec{x_n}]$  would be the matrix inverse of  $\vec{A}$ . In MATLAB, computation of the matrix inverse is invoked by the command  $\mathbf{inv}(\mathbf{A})$ .

Itisoftennecessarytocomputeadeterminantoraninverseinsomethingresemblingaclosedform(which will be seen in the calculation of *eigenvalues*). Thus, one introduces the *expansion by minors* method. A

minor  $M_{ij}(A)$  of a matrix A is the determinant of the matrix A without its  $i^{th}$  row and its  $j^{th}$  column.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$M_{12}(A) = \det \begin{bmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

A determinant is formed from expanding by minors along an arbitrary row i or column j of a matrix:

By Row *i*: 
$$det(A) = \sum_{j=1}^{n} a_{ij} M_{ij}(A) (-1)^{i+j}$$
  
By Column *j*:  $det(A) = \sum_{i=1}^{n} a_{ij} M_{ij}(A) (-1)^{i+j}$ 

Thus, what one has for an arbitrary matrix is a succession of expansions. First the matrix is broken down into a series of minors to determine the determinant, then these minors may need be broken down further into their minors to find their determinants, and so on until one gets to a  $1 \times 1$  matrix. For a scalar (i.e.,  $1 \times 1$ ) matrix,

$$A = [a_{11}] \Longrightarrow det(A) = a_{11}.$$

It is also easy to evaluate the determinant of a  $2 \times 2$  matrix by expanding along the first row,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Longrightarrow det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

With a little more effort, one can get the determinant for the case of a  $3 \times 3$  matrix. Here we expand along the first row.

$$A = \left[ \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right]$$

$$\implies det(A) = \begin{array}{c} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{array}$$

The inverse can also be found through expansion by minors. The recipe for this is a little more complicated than for the determinant. First one transposes the matrix, then each element  $a_{ij}$  gets replaced by the term  $M_{ij}(A)(-1)^{i+j}$ , where  $M_{ij}(A)$  is the minor of A at i and j. Finally, each resulting new element is divided by the determinant of the original matrix. For a 2 × 2 matrix, the inverse is

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Longrightarrow A^{-1} = in \, v(A) = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

The inverse of a  $3 \times 3$  matrix is somewhat more complicated.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad A' = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\implies A^{-1} = in \, v(A) = \frac{1}{det \, (A)} \begin{bmatrix} a_{22}a_{33} - a_{32}a_{23} & a_{32}a_{13} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{31}a_{23} - a_{21}a_{33} & a_{11}a_{33} - a_{31}a_{13} & a_{21}a_{13} - a_{11}a_{23} \\ a_{21}a_{32} - a_{31}a_{22} & a_{31}a_{12} - a_{11}a_{32} & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix}$$

Suppose that one applies this to the system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

where the solution  $\vec{x}$  is given by

$$\vec{x} = A^{-1} * \vec{b}$$

If one goes through the algebra, the result obtained from the Cramer's Rule will be

$$x_1 = \frac{1}{\det(A)} (b_1 a_{22} a_{33} + b_2 a_{32} a_{13} + b_3 a_{12} a_{23} - b_1 a_{32} a_{23} - b_2 a_{12} a_{33} - b_3 a_{22} a_{13})$$

$$\vdots = (\text{etc...})$$

This can be expressed more compactly as (with |A| being the shorthand notation for the determinant of A),

$$x_{1} = \frac{\begin{vmatrix} b_{1} & a_{12} & a_{13} \\ b_{2} & a_{22} & a_{23} \\ b_{3} & a_{32} & a_{33} \end{vmatrix}}{\det(A)}; \quad x_{2} = \frac{\begin{vmatrix} a_{11} & b_{1} & a_{13} \\ a_{21} & b_{2} & a_{23} \\ a_{31} & b_{3} & a_{33} \end{vmatrix}}{\det(A)}; \quad x_{3} = \frac{\begin{vmatrix} a_{11} & a_{12} & b_{1} \\ a_{21} & a_{22} & b_{2} \\ a_{31} & a_{32} & b_{3} \end{vmatrix}}{\det(A)};$$

### 2.3 Linear Ordinary Differential Equations

Note that in this course we encounter mostly linear ordinary differential equations with constant coefficients. One can always find an integrating factor for the first-order equation

$$\dot{x} + ax = f$$
,  $x(0^{-}) = x_0$ 

The term  $e^{at}$  is an integrating factor for the above equation. Let's multiply the above differential equation with the integrating factor  $e^{at}$  and collect terms, we have the following

$$\frac{d}{dt}\left(e^{at}x\right) = e^{at}f$$

Integrate both sides from  $0^-$  to time t,

$$x = e^{-at} \left[ x_o + \int_{0^-}^t e^{a\tau} f(\tau) d\tau \right]$$

or,

$$x = x_o e^{-at} + \int_{0^-}^t e^{-a(t-\tau)} f(\tau) d\tau$$

The above solution contains usually a homogeneous solution (from initial conditions) and a particular solution (from the forcing function f). Note that solution for the particular part involves a *convolution* integral.

A second-order equation given by

$$\ddot{x} + a\dot{x} + bx = f$$
,  $x(0^{-}) = x_0$ ,  $\dot{x}(0^{-}) = v_0$ 

can be written as a system of two first-order equations as follows. Let  $x_1 = x$  and  $x_2 = \dot{x}$ , the above equation can be re-written as

$$\left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ -b & -a \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] + \left[\begin{array}{c} 0 \\ f \end{array}\right]$$

In fact, the procedure leading to the solution of the above scalar first-order differential equation can be used to derive the solution for a system of first-order differential equations. In the latter case, it would involve the matrix exponential (i.e.,  $e^{At}$ ).

One could solve the above second-order system by first solving the *homogeneous* system whose solution is usually of the form  $x_h(t) = e^{\lambda t}$  where  $\lambda$  is the solution of the resulting *characteristic* equation. The particular solution from the *non-homogeneous* part can be done either by means of a trial substitution or variation of parameters. It turns out that the system of first-order differential equations is more amenable to use on the computer when expressed in a matrix state-space form. A formal way of modeling a dynamic system is by a set of state equations,

$$\dot{x} = Ax + Bu$$
 (State equations)  
 $y = Cx + Du$  (Output equations)

Theinput u is the particular forcing function driving the system (i.e., u = f). Refer back toour second-order system with both  $(x \text{ and } \dot{x})$  as outputs. Then, in the above notation we have the following set of system matrices

$$A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To generate a time series response of a linear time-invariant system inMATLAB, oneneeds to generate these A, B, C, and D matrices. Suppose that one creates a vector of time points where the system outputs are to be computed with the MATLAB command

#### T = [0:0.1:10];

This vector contains time points from 0 to 10 in steps of 0.1. The forcing function f (often referred to as the *control input*) is a vector whose entries correspond to the value of this function at those time points. To generate the system time responses for the system matrices A, B, C, and D as defined above, at the time points defined in the vector T to the forcing function f, we can issue the following MATLAB command

#### Y = lsim(A,B,C,D,f,T);

The vector Y has 2 columns (each corresponding to an output) and 101 rows (each row corresponding to a time point).

#### 2.4 Laplace Transform

Solving differential equations can also be done using the Laplace transform. We define

$$L(f(t)) = \int_{0^{-}}^{\infty} e^{-st} f(t) dt = f(s)$$

The function  $e^{-at}$  transforms as follows,

$$L(e^{-at}) = \int_{0^{-}}^{\infty} e^{-st} e^{-at} dt = \int_{0^{-}}^{\infty} e^{-(s+a)t} dt = -\frac{e^{-(s+a)t}}{s+a} \Big|_{0^{-}}^{\infty} = \frac{1}{s+a}.$$

Note that it is much easier to transform a function than to do its **inverse transform** which would involve from first principles the intricate details of complex variables and contour integration. However, it is much faster and easier for an engineer to use Table 2.1. Every function used in Laplace transform work is assumed multiplied by a unit-step (Heaviside) function at t = 0. That is to say, the function is identically zero for t < 0 and equals to one for  $t \ge 0$ .

| f(t)                                 | f(s)                                     |
|--------------------------------------|--|
| $\delta(t)$                          | 1  |
| 1 (unit step at t=0)                 | $\frac{1}{s}$                            |
| t                                    | $\frac{1}{s^2}$                          |
| $\frac{t^2}{2}$                      | $\frac{1}{s^3}$                          |
| $\frac{t^{n-1}}{(n-1)!}$             | $\frac{1}{s^n}$                          |
| $e^{\sigma t}$                       | $\frac{1}{s-\sigma}$                     |
| te <sup>σt</sup>                     | $\frac{1}{(s-\sigma)^2}$                 |
| $\frac{t^{n-1}e^{\sigma t}}{(n-1)!}$ | $\frac{1}{(s-\sigma)^n}$                 |
| $1 - e^{-\sigma t}$                  | $\frac{\sigma}{s(s+\sigma)}$             |
| $sin(\omega t)$                      | $\frac{\omega}{s^2+\omega^2}$            |
| $cos(\omega t)$                      | $\frac{s}{s^2+\omega^2}$                 |
| $e^{\sigma t} sin(\omega t)$         | $\frac{\omega}{(s-\sigma)^2+\omega^2}$   |
| $e^{\sigma t}cos(\omega t)$          | $\frac{s-\sigma}{(s-\sigma)^2+\omega^2}$ |

Table 2.1: Laplace Transforms of Some Common Functions

The Laplace transform is so attractive since

$$L(\frac{d}{dt}f(t)) = \int_{0^{-}}^{\infty} e^{-st} \frac{d}{dt}f(t) dt = e^{-st} f(t)|_{0^{-}}^{\infty} + s \int_{0^{-}}^{\infty} e^{-st} f(t) dt = -f(0^{-}) + sf(s)$$

Thus, one can transform a differential equation into a set of *algebraic* equations involving the transform variable s. For example, Laplace transform of the first-order differential equation

$$\dot{x} + ax = f(t)$$

is

$$sx(s) - x(0^{-}) + ax(s) = f(s)$$

Solving for the variable x(s), we obtain the solution of the above differential equation in the Laplace domain as

$$x(s) = \frac{x(0^{-})}{s+a} + \frac{f(s)}{s+a}$$

The first term of the sum corresponds to  $x(0^-)e^{-at}$ , the second is not solvable until one specifies f(t) (or f(s)). However, in general, the product of two Laplace transforms (in this case, f(s) and  $\frac{1}{s+a}$ ) is equivalent to the convolution of their time-based functions. This particular case is equal to what has been previously demonstrated,

$$L^{-1}\left\{\frac{f(s)}{s+a}\right\} = \int_{0^{-}}^{t} e^{-a(t-\tau)} f(\tau) d\tau$$

Consider now the second-order differential equation

$$\ddot{x} + a\dot{x} + bx = f$$

Applying Laplace transform to the above equation, we obtain

$$s^{2}x(s) - sx(0^{-}) - \dot{x}(0^{-}) + asx(s) - ax(0^{-}) + bx(s) = f(s)$$

and, solving for the solution x(s)

$$x(s) = \frac{\dot{x}(0^{-}) + (s+a)x(0^{-}) + f(s)}{s^{2} + as + b}$$

The homogeneous parts of this have equivalent time-domain functions that depend on the relation between *a* and *b*. We distinguish three cases:

- $b a^2/4 > 0$
- $b a^2/4 < 0$
- $b a^2/4 = 0$

Case  $b - a^2/4 > 0$ : The denominator can be written into the form  $s^2 + 2\sigma s + \sigma^2 + \omega^2$  which corresponds to the time-domain functions  $e^{-\sigma t} \sin \omega t$  and  $e^{-\sigma t} \cos \omega t$  where  $\sigma = -a/2$  and  $\omega = \sqrt{b - a^2/4}$ . Solutions to the homogeneous problem (i.e., to initial conditions  $x(0^-)$ ) can be obtained directly as

$$x_h(t) = \dot{x}(0^-) \frac{e^{-at/2} \sin(\sqrt{b - a^2/4} t)}{\sqrt{b - a^2/4}} + x(0^-) e^{-at/2} \left\{ \cos(\sqrt{b - a^2/4} t) + \frac{a}{2\sqrt{b - a^2/4}} \sin(\sqrt{b - a^2/4} t) \right\}$$

Use of Table 2.1 is not always possible with some more complicated forms. Usually one needs to break down a complicated polynomial fraction into simpler summands that are of the forms given in Table 2.1.

Suppose that the forcing function f(t) is a step input (applied at t=0) whose Laplace transform is simply 1/s. We will derive the particular solution as an illustration to the *partial fraction expansion* methodology. The particular solution to the non-homogenous problem is

$$x_p(s) = \frac{1}{s(s^2 + as + b)}$$

The right-hand term can be decomposed into

$$\frac{1}{s(s^2 + as + b)} = \frac{u}{s} + \frac{vs + w}{s^2 + as + b}$$

with unknowns u, v and w. Expanding and matching the numerator term, we have

$$u(s^2 + as + b) + vs^2 + ws = 1$$

Since the coefficients of  $s^2$  and s must be zero, and that ub = 1, we have

$$u = 1/b$$
 ,  $v = -1/b$  ,  $w = -a/b$ 

The particular solution is given by

$$x_p(t) = \frac{1}{b} \left[ 1 - e^{-at/2} \left\{ \cos(\sqrt{b - a^2/4} t) + \frac{a}{2\sqrt{b - a^2/4}} \sin(\sqrt{b - a^2/4} t) \right\} \right]$$

Case  $b - a^2/4 < 0$ : In this case, we have two distinct real roots to the equation  $s^2 + as + b = 0$ . They are given by

$$\sigma_1 = -a/2 + \sqrt{a^2/4 - b}$$

$$\sigma_2 = -a/2 - \sqrt{a^2/4 - b}$$

Solution to the homogenous problem is simply

$$x_h(s) = \frac{\dot{x}(0^-) + (s+a)x(0^-)}{(s-\sigma_1)(s-\sigma_2)}$$

or,

$$x_h(t) = \frac{\dot{x}(0^-) + (\sigma_1 + a)x(0^-)}{\sigma_1 - \sigma_2} e^{\sigma_1 t} + \frac{\dot{x}(0^-) + (\sigma_2 + a)x(0^-)}{\sigma_2 - \sigma_1} e^{\sigma_2 t}$$

Similarly, for the particular solution we have

$$x_p(s) = \frac{1}{s(s^2 + as + b)}$$

In partial fraction expansion

$$x_p(s) = \frac{u}{s} + \frac{v}{s - \sigma_1} + \frac{w}{s - \sigma_2}$$

The unknowns u, v and w are determined from the following equation

$$u(s^2 + as + b) + vs^2 - vs\sigma_2 + ws^2 - ws\sigma_1 = 1$$

For the coefficients of  $s^2$  and s to vanish we must have u + v + w = 0 and  $ua - v\sigma_2 - w\sigma_1 = 0$  with ub = 1. Thus, we have

$$u = \frac{1}{b} = \frac{1}{\sigma_1 \sigma_2}$$
,  $v = \frac{1}{\sigma_1 (\sigma_1 - \sigma_2)}$ ,  $w = \frac{1}{\sigma_2 (\sigma_2 - \sigma_1)}$ 

Hence, the time-domain solution is

$$x_p(t) = \left(\frac{1}{\sigma_1 \sigma_2} + \frac{1}{\sigma_1(\sigma_1 - \sigma_2)} e^{\sigma_1 t} + \frac{1}{\sigma_2(\sigma_2 - \sigma_1)} e^{\sigma_2 t}\right) H(t)$$

where H(t) corresponds to the Heaviside (step) function at t = 0.

Case  $a^2/4 = b$ : This case is similar to the previous case where

$$\sigma_2 = \sigma_1 = -\frac{a}{2}$$

Solution to the homogenous problem is simply

$$x_h(s) = \frac{\dot{x}(0^-) + (s+a)x(0^-)}{(s-\sigma_1)^2}$$

or,

$$x_h(t) = \dot{x}(0^-)te^{\sigma_1 t} + x(0^-)(1 - \sigma_1 t)e^{\sigma_1 t}$$

Similarly, for the particular solution we have

$$x_p(s) = \frac{1}{s(s^2 + as + b)} = \frac{u}{s} + \frac{v}{s - \sigma_1} + \frac{w}{(s - \sigma_1)^2}$$

For the coefficients of  $s^2$  and s to vanish we must have u + v = 0 and  $-2\sigma_1 u - \sigma_1 v + w = 0$  with u = 1/b. Hence,

$$u = \frac{1}{\sigma_1^2}$$
 ,  $v = -\frac{1}{\sigma_1^2}$  ,  $w = \frac{1}{\sigma_1}$ 

Or, in the time domain

$$x_p(t) = \left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_1^2}e^{\sigma_1 t} + \frac{1}{\sigma_1}te^{\sigma_1 t}\right)H(t)$$

In MATLAB, time responses of a system can be obtained from their Laplace transforms directly. Response of the outputs to an input U defined over the time points T can be obtained using the command

#### Y = lsim(NUM,DEN,U,T);

where the arguments **NUM** and **DEN** are arrays containing coefficients of the numerator and denominator polynomials in s arranged in descending powers of s.

### 2.5 Stability

Dynamicstability ischaracterized by the response of a system to nonzero initial conditions. Initial conditions are equivalent to an impulsive forcing function (i.e. Dirac delta function  $u(t) = \delta(t)$ ).

For a first-order system  $\dot{x} + ax = 0$  and  $x(0^-) = x_o$ , we have the homogeneous solution  $x(t) = x_o e^{-at}$ . It is simple to imagine that for a > 0, the response x(t) to initial conditions  $x_o$  would tend toward zero (thus stable) as  $t \to \infty$ . On the other hand, if a < 0 the response would tend to blow up.

For a second-order system  $\ddot{x} + a\dot{x} + bx = 0$ , we have solutions of the form

2.6. EXAMPLE 17

• When  $a^2/4 < b$ ,

$$x(t) = e^{\sigma t} \left[ u \sin \omega t + v \cos \omega t \right]$$

• When  $a^2/4 > b$ ,

$$x(t) = ue^{\sigma_1 t} + ve^{\sigma_2 t}$$

• When  $a^2/4 = b$ ,

$$x(t) = ue^{\sigma t} + vte^{\sigma t}$$

In any case, the argument  $\sigma$  in the exponential function  $e^{\sigma t}$  term must be less than or equal to zero. In the case of  $a^2/4 > b$ , both terms  $\sigma_1$  and  $\sigma_2$  must be less than or equal to zero. Otherwise, the solution will blow up as  $t \to \infty$ . For  $\sigma$  equal to zero, then one has a *neutrally stable* system.

#### 2.6 Example

Consider the following ordinary differential equation

$$\frac{d^3y(t)}{dt^3} + 5\frac{d^2y(t)}{dt^2} + 17\frac{dy(t)}{dt} + 13y(t) = 13u(t)$$
 (2.1)

with initial conditions  $y(0^-) = 7$ ,  $\dot{y}(0^-) = 0$  and  $\ddot{y}(0^-) = 0$ . Solve for the time response y(t) when  $u(t) = \delta(t)$  (impulse function or Dirac delta function).

We can solve the problem using three methods:

- Laplace method
- Time-domain method involving the matrix exponential
- Numerical integration method (via MATLAB)

#### 2.6.1 Laplace method

Taking the Laplace transform on the differential equation, we obtain

$$\left[ s^{3}y(s) - s^{2}y(0^{-}) - s\dot{y}(0^{-}) - \ddot{y}(0^{-}) \right] + 5\left[ s^{2}y(s) - sy(0^{-}) - \dot{y}(0^{-}) \right] +$$

$$17\left[ sy(s) - y(0^{-}) \right] + 13y(s) = 13u(s)$$
(2.2)

With  $y(0^-) = 7$ ,  $\dot{y}(0^-) = \ddot{y}(0^-) = 0$ , equation 2.2 becomes

$$[s^{3} + 5s^{2} + 17s + 13]y(s) - [s^{2} + 5s + 17]y(0^{-}) = 13u(s)$$
(2.3)

or, solving for y(s) we have

$$y(s) = \frac{s^2 + 5s + 17}{s^3 + 5s^2 + 17s + 13}y(0^-) + \frac{13}{s^3 + 5s^2 + 17s + 13}u(s)$$
 (2.4)

For a unit impulse input  $u(t) = \delta(t)$  or u(s) = 1 and  $y(0^-) = 7$ , equation 2.4 becomes

$$y(s) = \frac{s^2 + 5s + 17}{s^3 + 5s^2 + 17s + 13} 7 + \frac{13}{s^3 + 5s^2 + 17s + 13} 1$$

$$= \frac{7s^2 + 35s + 132}{(s+1)[(s+2)^2 + 3^2]}$$
(2.5)

Performing the partial fraction expansion on equation 2.5, we have

$$y(s) = \frac{R_1}{s+1} + \frac{R_2}{s+2-j3} + \frac{R_3}{s+2+j3}$$
 (2.6)

where

$$R_{1} = \frac{7s^{2} + 35s + 132}{\left[(s+2)^{2} + 3^{2}\right]} \Big|_{s=-1} = 10.4$$

$$R_{2} = \frac{7s^{2} + 35s + 132}{(s+1)(s+2+j3)} \Big|_{s=-2+j3} = -1.7 - j0.6$$

$$R_{3} = \frac{7s^{2} + 35s + 132}{(s+1)(s+2-j3)} \Big|_{s=-2-j3} = -1.7 + j0.6 = \bar{R}_{2}$$

where  $\bar{R}_2$  is the complex conjugate of  $R_2$ . Taking the inverse Laplace transform on equation 2.6, we have

$$y(t) = R_1 e^{-t} + R_2 e^{(-2+j3)t} + \bar{R}_2 e^{(-2-j3)t}$$
(2.7)

or since  $e^{a+jb} = e^a(cosb + jsinb)$ , we have

$$y(t) = 10.4e^{-t} + e^{-2t}(-1.7 - j0.6)(\cos 3t + j\sin 3t) + (-1.7 + j0.6)(\cos 3t - j\sin 3t)$$
 (2.8)

$$y(t) = 10.4e^{-t} + 2e^{-2t}(-1.7\cos 3t + 0.6\sin 3t)$$
(2.9)

#### 2.6.2 Time-domain method

Let's define  $x_1(t) = y(t)$ ,  $x_2(t) = \dot{y}(t)$  and  $x_3(t) = \ddot{y}(t)$ , then clearly

$$\dot{x}_1(t) = x_2(t) = \dot{y}(t) \tag{2.10}$$

$$\dot{x}_2(t) = x_3(t) = \ddot{y}(t)$$
 (2.11)

and equation 2.1 can be re-written as

$$\dot{x}_3(t) + 5x_3(t) + 17x_2 + 13x_1(t) = 13u(t) \tag{2.12}$$

Combining equations 2.10-2.12 into a set of three first-order differential equations which can be expressed in a matrix equation,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -13 & -17 & -5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 13 \end{bmatrix} u(t)$$
 (2.13)

2.6. EXAMPLE

with initial conditions  $x_1(0^-) = y(0^-)$ ,  $x_2(0^-) = \dot{y}(0^-)$  and  $x_3(0^-) = \ddot{y}(0^-)$ . Equation 2.13 can be written in a concise form as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0^{-}) = x_o \end{cases}$$
 (2.14)

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -13 & -17 & -5 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 13 \end{bmatrix}$$

$$x_o = \begin{bmatrix} y(0^-) \\ \dot{y}(0^-) \\ \ddot{y}(0^-) \end{bmatrix}$$

Solution of equation 2.13 is obtained using the method of linear superposition and convolution (Duhamel) integral. Namely,

$$x(t) = e^{At}x_0 + \int_{0^-}^t e^{A(t-\tau)}Bu(\tau)d\tau$$
 (2.15)

Since  $u(t) = \delta(t)$  equation 2.15 becomes

$$x(t) = e^{At}x_o + \int_{0^-}^t e^{A(t-\tau)}B\delta(\tau)d\tau$$
 (2.16)

or

$$x(t) = e^{At}x_o + e^{At}B = e^{At}(x_o + B)$$
(2.17)

Since  $x_o = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$ , we have

$$x(t) = e^{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -13 & -17 & -5 \end{bmatrix}} t \begin{pmatrix} \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 13 \end{bmatrix}$$
 (2.18)

or

$$x(t) = e^{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -13 & -17 & -5 \end{bmatrix}} t \begin{bmatrix} 7 \\ 0 \\ 13 \end{bmatrix}$$
 (2.19)

#### 2.6.3 Numerical integration method (via MATLAB)

In MATLAB, the command **lsim** would perform numerical integration on the linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$
 (2.20)

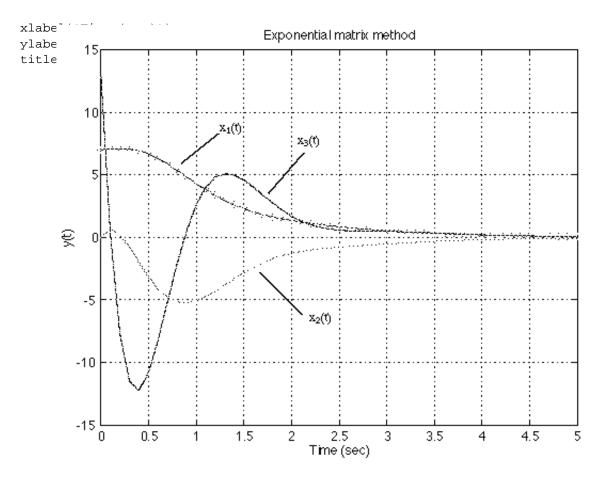
for a given initial condition  $x(0^-) = x_o$  and an input function u(t) defined in the time interval  $0 \le t \le t_{max}$ . More precisely, we can use the following set of MATLAB codes to perform the time responses of y(t) for the linear system described in equation 2.20.

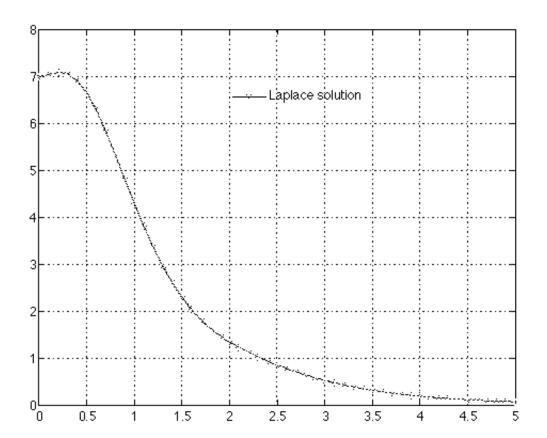
```
t=[0:.1:tmax];
u= "some function of t" (e.g., u=ones(t) for a step input)
y=lsim(A,B,C,D,u,t,xo)
plot(t,y)
```

Below is a complete listing of the m-file for this example problem.

```
%Using the Laplace method
t=[0:.1:5];
y=10.4*exp(-t)+2*exp(-2*t) .* (-1.7*cos(3*t)+0.6*sin(3*t));
%Using the state-space method
A=[0,1,0;0,0,1;-13,-17,-5];
B=[0;0;13];
C=[1,0,0];
D=0;
xo=[7;0;0];
[n,m]=size(t);
%Solution from the exponential matrix
yexp=[];
for i=1:length(t)
ti=t(i);
ytemp= expm(A*ti)*(xo+B);
yexp=[yexp;ytemp'];
end
%MATLAB command for solving time responses
%For an impulse input the new initial condition becomes x(o+)=x(0-)+b
% and the input u(t)=0 for t>0+
xoplus=xo+B;
u=zeros(n,m);
ysim=lsim(A,B,C,D,u,t,xoplus);
%Plot the responses for comparison
plot(t,y,'o',t,yexp)
grid
xlabel('Time (sec)')
ylabel('y(t)')
title('Exponential matrix method')
pause
%Plot the responses for comparison
plot(t,y,'o',t,ysim)
grid
```

2.6. EXAMPLE 21





## **Chapter 3**

## **Principles of Static and Dynamic Stability**

In most design situation, static and dynamic stability analysis plays a significant role in the determination of the final airplane design configuration. The decision is based according to the requirements defined in FAR Part 23 which states that

"the airplanemust besafely controllableand maneuverableduring — (1)take off; (2)climb; (3)levelflight; (4)dive; and (5)landing (poweron, off) (with wingflaps extended and retracted)"

Stability of such a vehicle is also a major consideration in selecting a particular design configuration. The airplane must be longitudinally, directionally and laterally stable for airworthiness and minimal pilot workload. If the airplane turns out to have undesireable flying qualities, then some of these requirements must subsequently be met by the use of stability augmentation systems. This requires careful design of a control system that feedbacks sensed aircraft motion variables to the appropriate control surfaces (e.g. elevator, aileron and rudder). The topic of feedback synthesis of flight control systems for stability augmentation and autopilot designs is the subject of AA-517 and a continuation in AA-518. In the present course, we will only examine the fundamental behaviour of flight vehicle and its inherent flight characteristics without the influence of *artificial* feedback control.

The general notion of stability refers to the tendency of the vehicle to return to its original state of equilibrium (e.g., trim point) when disturbed. There are basically two types of stability:

• Static stability refers to the tendency of an airplane under static conditions to return to its trimmed condition. Clearly we assume that there exists an equilibrium point about which static stability is investigated. The evaluation of static stability involves purely static (i.e. steady-state) equations from force and moment balance applied to a vehicle disturbed from its equilibrium. Conditions for stability are governedby the direction of the forces andmoments that will restore the vehicle to the original trim states. Figure 3.1 shows the three possible cases of static stability. Clearly, from these illustrations, we determine stability from the direction of the restoring force. In Figure 3.1 (a) component of the gravity force tangential to the surface will bring the ball back to its original equilibrium point. However, in static stability analysis there is nomention on how and when the ball will return to its equilibrium point. For example, without the benefit of friction, the ball will oscillate back and forth about the equilibrium point and therefore will never reach the equilibrium state. To treat this problem concerning the dynamic behaviour of this ball rolling on a curved surface, one will need to first develop its equation of motion and then analyze the stability of its motion when released from a perturbed position. Stability of the ball in motion is then determined by the phenomena of dynamic stability.

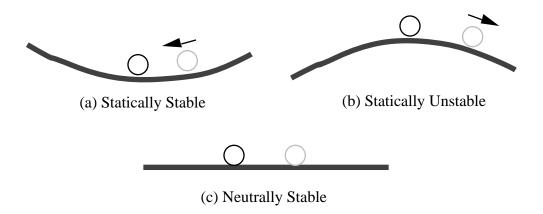


Figure 3.1: Three Possible Cases of Static Stability

• Dynamicstability isgovernedbythefactthevehiclewillreturntoitsoriginalequilibriumconditionafter some interval of time. As discussed in the previous section, analysis of dynamic stability would entail a complete modeling of the vehicle dynamics and its responses when perturbed from the equilibrium state. Figure 3.2 shows typical responses of a dynamically stable, unstable and neutrally stablesystem. It is important to observe from the above examples that a dynamically stable airplane must always be statically stable. On the other hand, a statically stable airplane is not necessary dynamically stable. Detailed study of dynamic stability of a flight vehicle will be performed following the development of the general equations of motion of a rigid-body airplane in Chapter 7.

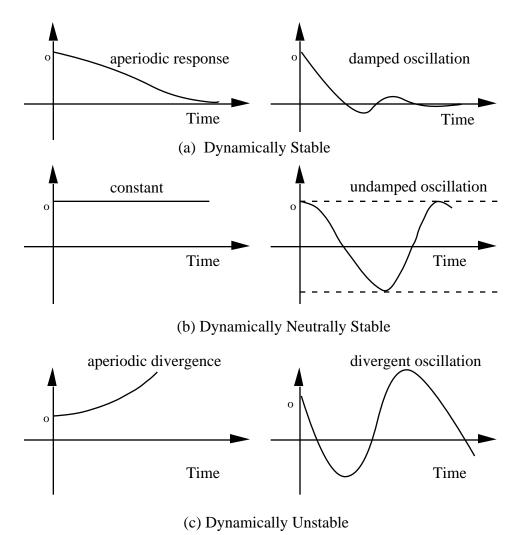


Figure 3.2: Three Possible Cases of Dynamic Stability

## **Chapter 4**

## **Static Longitudinal Stability**

A study of airplane stability and control is primarily focused on moments about the airplane center of gravity. A balanced (i.e., trimmed) airplane will have zero moment about its center of gravity. The total moment coefficient about the center of gravity is defined as

$$C_{M_{cg}} = \frac{M_{cg}}{qSc} \tag{4.1}$$

where S is the wing planform area, c is the mean aerodynamic chord and  $q_{\infty}$  is the dynamic pressure corresponding to the freestream velocity  $V_{\infty}$ . There are numerous places where moments can be generated in an airplane (Figure 4.1) such as moments contributed by the wing, the fuselage, the engine propulsion, the controls (e.g., elevator, aileron, rudder, canard, etc...) and the vertical and horizontal tail surfaces. Note that the gravity force does not contribute any moment to the airplane since it is, by definition, applied at the center of gravity. The *aerodynamic center* for the wing is *defined* as the point about which the moment  $M_{ac}$  (or its moment coefficient  $C_{M,ac}$ ) is independent of the angle of attack. This point is convenient for the derivation of the moment equation since it *isolates* out the part that is independent of the angle of attack.

### 4.1 Notations and Sign Conventions

Hereweintroducethecommonlyusednotationsfordisplacements, velocities, forces and moments instability, control and flight mechanics. The origin of the axis system defined by the x, y, z-coordinates is assumed fixed to the center of gravity of the airplane (see Figure 4.2). It will move and rotate with the aircraft. The x displacement has a positive forward direction, the y displacement has a positive direction to the right-wing direction while the z displacement pointed positively downward. Therespective components of the aircraft velocity  $\mathbf{V}$  in the x, y and z directions are (u, v, w) respectively. The total force  $\mathbf{F}$  applied to the airplane has components (X, Y, Z) while the respective moment components are (L, M, N). Note that all the forces and moments are assumed to apply at the center of gravity.

We will examine a simple airplane configuration in our analysis of longitudinal static stability. The basic airplane consists simply of a wing and tail configuration *only*. This simple configuration will illustrate well the basic fundamentals in stability and control analysis.

## 4.2 Stick-Fixed Stability

The forces and moments of a wing-tail configuration is shown in Figure 4.3. Without loss of generality, the

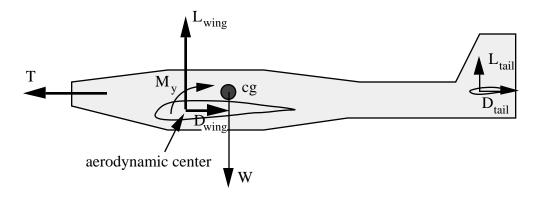


Figure 4.1: Moments about the Center of Gravity of the Airplane

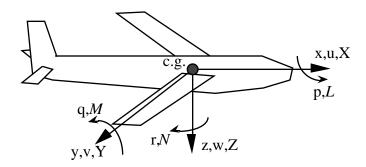


Figure 4.2: Definition of Aircraft Variables in Flight Mechanics

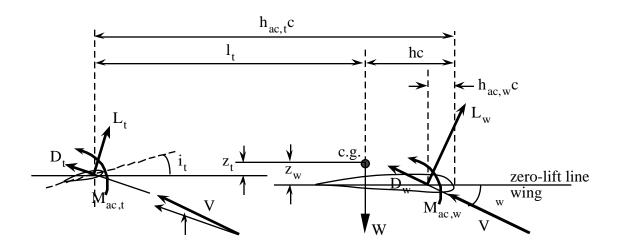


Figure 4.3: Forces and Moments Applied to a Wing-Tail Configuration

horizontal axis is assumed to coincide with the zero-lift line of the wing. Relative to this reference line, the tail is shown to have a positive incidence angle. Note that in our development, we adopt the same standard convention for all angle definitions (i.e. according to the right-hand rule). The angle of attack of the wing with respect to the zero-lift line is defined as  $\alpha_w$ . At the tail, the angle of attack is reduced by an angle  $\epsilon$  due to the downwash at the wing. The airplane is in equilibrium when sums of all the forces and moments about the center of gravity are zero.

In the longitudinal axis, we have

$$\Sigma F_z = W - (L_w \cos \alpha_w + D_w \sin \alpha_w) - [L_t \cos (\alpha_w - \epsilon) + D_t \sin (\alpha_w - \epsilon)]$$

$$= 0$$
(4.2)

and

$$\Sigma M_{y} = M_{ac,w} + (L_{w}cos\alpha_{w} + D_{w}sin\alpha_{w})(hc - h_{ac,w}c) + (L_{w}sin\alpha_{w} - D_{w}cos\alpha_{w})z_{w} + M_{ac,t} - [L_{t}cos(\alpha_{w} - \epsilon) + D_{t}sin(\alpha_{w} - \epsilon)]l_{t} + [L_{t}sin(\alpha_{w} - \epsilon) - D_{t}cos(\alpha_{w} - \epsilon)]z_{t}$$

$$= 0$$

$$(4.3)$$

To simplify our analysis, one can usually assume that the angle of attack  $\alpha_w$  is small and use the following approximations for  $\cos \alpha_w \cong 1$  and  $\sin \alpha_w \cong \alpha_w$  where  $\alpha_w$  is in radians. Then equations (4.2) and (4.3) become

$$W = (L_w + D_w \alpha_w) + [L_t + D_t (\alpha_w - \epsilon)] \tag{4.4}$$

and

$$M_{ac,w} + (L_w + D_w \alpha_w)(h - h_{ac,w})c + (L_w \alpha_w - D_w)z_w + M_{ac,t} - [L_t + D_t(\alpha_w - \epsilon)]l_t + [L_t(\alpha_w - \epsilon) - D_t]z_t = 0$$
(4.5)

Let's introduce the following definitions for non-dimensional forces and moments at the wing and tails urfaces,

$$L_{w} = qS \frac{dC_{Lw}}{d\alpha} \alpha_{w}$$

$$= qSa_{w} \alpha_{w}$$

$$M_{ac,w} = qScC_{M_{ac,w}}$$

$$L_{t} = q_{t}S_{t}C_{L_{t}}$$

$$= q_{t}S_{t}\frac{dC_{Lt}}{d\alpha}\alpha_{t}$$

$$= q_{t}S_{t}\frac{dC_{Lt}}{d\alpha}(i_{t} + \alpha_{w} - \epsilon)$$

$$= q_{t}S_{t}a_{t}(i_{t} + \alpha_{w} - \epsilon)$$

$$M_{ac,t} = q_{t}S_{t}c_{t}C_{M_{ac,t}}$$

$$(4.6)$$

Furthermore, we define the following

$$\eta_t = \frac{q_t}{q} \tag{4.7}$$

$$\epsilon = \epsilon_o + \frac{d\epsilon}{d\alpha} \alpha_w 
= \epsilon_o + \epsilon_\alpha \alpha_w$$
(4.8)

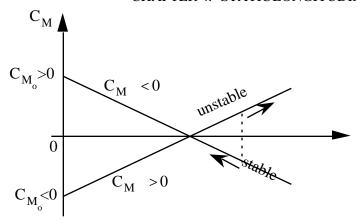


Figure 4.4: Moment Coefficient  $C_{M_{cg}}$  versus  $\alpha$ 

where  $\epsilon_o$  is the downwash angle when the wing is at zero lift. Both  $\epsilon_o$  and  $\epsilon_\alpha$  are obtained from wind tunnel data. And the variable  $\eta_t$  is simply the ratio of dynamic pressure at the tail to the freestream dynamic pressure; this can be greater or less than unity depending on whether the tail is in the wake of the propulsion system or not.

Further simplification can be obtained using the fact that  $D_w \alpha_w \ll L_w$ ,  $D_t(\alpha_w - \epsilon) \ll L_t$ ,  $z_w \cong 0$  and  $z_t \cong 0$ . Equations (4.4) and (4.5) simplify to

$$C_{L} = \frac{W}{qS}$$

$$= a_{w}\alpha_{w} + \eta_{t} \frac{S_{t}}{S} a_{t} (i_{t} + \alpha_{w} - \epsilon)$$

$$= C_{L_{o}} + C_{L_{\alpha}} \alpha_{w}$$

$$= C_{L_{\alpha}} (\alpha_{w} - \alpha_{w,zerolift})$$

$$(4.9)$$

where

$$C_{L_o} = \eta_t \frac{S_t}{S} a_t (i_t - \epsilon_o)$$

$$C_{L_\alpha} = a_w + \eta_t \frac{S_t}{S} a_t (1 - \epsilon_\alpha)$$
(4.10)

and

$$C_{M_{cg}} = C_{M_{ac,w}} + a_{w}\alpha_{w}(h - h_{ac,w}) + \frac{q_{t}S_{t}c_{t}}{qSc}C_{M_{ac,t}} - \frac{q_{t}S_{t}l_{t}}{qSc}a_{t}(i_{t} + \alpha_{w} - \epsilon) + C_{M_{\alpha fus}}\alpha_{w}$$

$$= C_{M_{ac,w}} + \frac{q_{t}S_{t}c_{t}}{qSc}C_{M_{ac,t}} - \eta_{t}V_{H}a_{t}(i_{t} - \epsilon_{o}) + \{(h - h_{ac,w})a_{w} - \eta_{t}V_{H}a_{t}(1 - \epsilon_{\alpha}) + C_{M_{\alpha fus}}\}\alpha_{w}$$

$$= C_{M_{o}} + C_{M_{\alpha}}\alpha_{w}$$

(4.11)

where

$$C_{M_o} = C_{M_{ac,w}} + \frac{q_t S_t c_t}{q Sc} C_{M_{ac,t}} - \eta_t V_H a_t (i_t - \epsilon_o)$$

$$C_{M_\alpha} = (h - h_{ac,w}) a_w - \eta_t V_H a_t (1 - \epsilon_\alpha) + C_{M_{\alpha fus}}$$

$$= \frac{dC_M}{dC_L} C_{L_\alpha}$$

$$(4.12)$$

 $C_{M_{\alpha fus}}$  identifies the contribution of the fuselage to the pitchingmoment (it is generally negligeable  $C_{M_{\alpha fus}} \approx 0$ ), and  $V_H = \frac{S_t l_t}{S_C}$  is the horizontal tail volume coefficient. Recall that in equilibrium, we must have  $C_{M_{cg}} = 0$ . Referring to Figure 4.4, one can see that there are two possible cases for an equilibrium to exist; namely,

- 1.  $C_{M_o} > 0$  and  $C_{M_\alpha} < 0$ : This case corresponds to a statically stable equilibrium point since for any small change in the angle of attack, a restoring moment generated to bring it back to the equilibrium.
- 2.  $C_{M_o} < 0$  and  $C_{M_\alpha} > 0$ : This case corresponds to a statically unstable equilibrium point since the moment created due to any change in angle of attack will tend to increase it further.

There exists a location of the center of gravity, i.e. when  $h = h_n$ , where the coefficient  $C_{M_{\alpha}} = 0$ . Recall that  $V_H = \frac{S_t l_t}{S_C}$  (Tail volume coefficient) and  $l_t = (h_{ac,t} - h)c$ , then equation (4.12) becomes, with h substituted by  $h_n$ .

$$(h_n - h_{ac,w})a_w - \eta_t (h_{ac,t} - h_n) \frac{S_t}{S} a_t (1 - \epsilon_\alpha) + C_{M_{\alpha fus}} = 0$$
 (4.13)

or

$$[a_w + \eta_t \frac{S_t}{S} a_t (1 - \epsilon_\alpha)] h_n = h_{ac, w} a_w + \eta_t \frac{S_t}{S} a_t (1 - \epsilon_\alpha) h_{ac, t} - C_{M_{\alpha fus}}$$

$$\tag{4.14}$$

Let's examine the total lift on the wing-tail configuration, it is given by

$$L = L_w + L_t$$

$$= qSa_w\alpha_w + q_tS_ta_t(i_t + \alpha_w - \epsilon)$$

$$= qSa_w\alpha_w + q_tS_ta_t\{i_t - \epsilon_\alpha + \alpha_w(1 - \epsilon_\alpha)\}$$
(4.15)

or the total lift coefficient  $C_L$  is

$$C_{L} = \eta_{t} \frac{S_{t}}{S} a_{t} (i_{t} - \epsilon_{o}) + [a_{w} + \eta_{t} \frac{S_{t}}{S} a_{t} (1 - \epsilon_{\alpha})] \alpha_{w}$$

$$= C_{L_{o}} + C_{L_{\alpha}} \alpha_{w}$$
(4.16)

Thus, the combined lift curve slope is

$$C_{L_{\alpha}} = a_w + \eta_t \frac{S_t}{S} a_t (1 - \epsilon_{\alpha}) \tag{4.17}$$

From the above definition of  $C_{L_{\alpha}}$ , equation (4.14) is simplified to the following

$$C_{L_{\alpha}}h_{n} = h_{ac,w}a_{w} + (C_{L_{\alpha}} - a_{w})h_{ac,t} - C_{M_{\alpha}f_{us}}$$
(4.18)

or the neutral point  $h_n$  is given by

$$h_n = \frac{h_{ac,w} + \left[\frac{C_{L_{\alpha}}}{a_w} - 1\right]h_{ac,t}}{\frac{C_{L_{\alpha}}}{a}} - \frac{C_{M_{\alpha fus}}}{C_{L_{\alpha}}}$$
(4.19)

or

$$h_n = h_{ac,t} - \frac{a_w}{C_{L_{\alpha}}}(h_{ac,t} - h_{ac,w}) - \frac{C_{M_{\alpha fus}}}{C_{L_{\alpha}}}$$

From the above, it can be easily shown that

$$C_{M_{\alpha}} = C_{L_{\alpha}}(h - h_n) \tag{4.20}$$

Note that  $C_{L_{\alpha}} > 0$ , thus  $C_{M_{\alpha}} < 0$  if  $(h - h_n) < 0$  or, the center of gravity must be ahead of the neutral point. The other condition  $C_{M_o} > 0$ , where  $C_{M_o}$  is defined in equation (4.12), will be satisfied if the tail incidence angle  $i_t$  is negative. The quantity  $(h_n - h)$  is called the *static margin*. It represents the distance (expressed as a fraction of the mean aerodynamic chord) that the center of gravity is ahead of the neutral point. Roughly, a desireable static margin of at least 5% is recommended. For airplane with *relaxed* static stability, the static margin is negative. A stability augmentation system (SAS) is needed to fly these vehicle.

### Example 1

Given a light airplane with the following design parameters,

- Wing area  $S_w = 160.0 ft^2$ , Wing span  $b_w = 30 ft$ ,
- Horizontal tail area  $S_t = 24.4 ft^2$ , Tail span  $b_t = 10 ft$ ,
- $h_{ac,t} = 2.78$
- Wing with  $65_2 415$  type airfoil,  $C_{M_{ac,w}} = -0.07$ ,  $h_{ac,w} = 0.27$ ,
- $\eta_t \cong 1$  and  $\epsilon_\alpha \cong 0.447$ .

The lift curve slopes at the wing and tail are obtained from the following empirical equation,

$$a_{3D} = a_{2D} \frac{A}{A + [2(A+4)/(A+2)]}$$
(4.21)

where  $A = b^2/S$  is the aspect ratio of the surface and no sweep. Thus, with  $a_{2D} = 2\pi$  per radians = 0.106 per degrees, we have

$$a_w = 0.0731 \text{per degrees}$$
  
 $a_t = 0.0642 \text{per degrees}$  (4.22)

The total lift curve slope according to equation (4.17) is  $C_{L_{\alpha}} = 0.0785$  per degrees, and the neutral point is at  $h_n = 0.443$ . Then

$$C_{M_{\alpha}} = 0.0785(h - 0.443) \tag{4.23}$$

# Calculation of $C_{M_{ac}}$ , Aerodynamic Center Location and Mean Aerodynamic Chord (mac) for a Finite Wing

Consider a finite wing shown in Figure 4.5. The locus of the section aerodynamic centers defines the swept back angle  $\Lambda$ . The pitching moment about a line through the point A and normal to the chord line is given by

$$M_A = q \int_{-h/2}^{b/2} c^2 C_{m_{ac}} dy - q \int_{-h/2}^{b/2} c C_l y \tan \Lambda dy$$
 (4.24)

Then if  $X_A$  is the distance of the aerodynamic center behind the point A, then

$$M_{ac} = M_A + LX_A \tag{4.25}$$

or

$$C_{M_{ac}} = C_{M_A} + C_L \frac{X_A}{\bar{c}} \tag{4.26}$$

Differentiating  $C_{M_{ac}}$  with respect to  $\alpha$  and using the definition of an aerodynamic center yields

$$0 = \frac{dC_{M_A}}{d\alpha} + \frac{X_A}{\bar{c}} C_{L_\alpha} \tag{4.27}$$

Substituting equation (4.24) into the above equation and using the fact that

$$\frac{dC_{m_{ac}}}{d\alpha} = 0 (4.28)$$

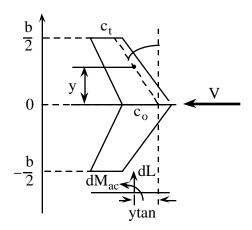


Figure 4.5: Calculation of Wing Aerodynamic Center

we obtain from equation (4.27),

$$X_A = \frac{1}{C_{L_{\alpha}} S} \int_{-b/2}^{b/2} c C_{l_{\alpha}} y \tan \Lambda dy$$
 (4.29)

If we assume that  $C_{l_{\alpha}}$  is constant across the wing span, then we have

$$X_A = \left[\frac{\int_0^{b/2} cy dy}{S/2}\right] \frac{C_{l_\alpha}}{C_{L_\alpha}} \tan \Lambda \tag{4.30}$$

or

$$X_A = \bar{y} \frac{C_{l_{\alpha}}}{C_{I_{\alpha}}} \tan \Lambda \tag{4.31}$$

where  $\bar{y}$  is the spanwise distance from the centerline out to the centroid of the half-wing area. As a special case, for a linearly tapered wing, equation (4.31) becomes

$$X_A = \frac{(1+2\lambda)}{(1+\lambda)} \frac{C_{l_\alpha}}{C_{L_\alpha}} \frac{b}{6} \tan \Lambda \tag{4.32}$$

where  $\lambda = \frac{c_t}{c_o}$  is the wing taper ratio. The mean aerodynamic chord  $\bar{c}$  of a finite wing is defined as the chord length that, when multiplied by the wing area S, the dynamic pressure q, and an average  $C_{M_{ac}}$ , gives the total moment about the wing's aerodynamic center. Namely,

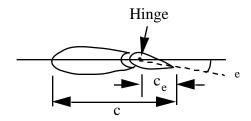
$$M_{ac} = qS\bar{c}C_{M_{ac}} \tag{4.33}$$

Combining the above equation with equation (4.24), we have

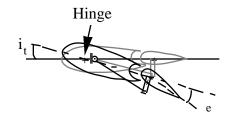
$$qS\bar{c}C_{M_{ac}} = q \int_{-b/2}^{b/2} c^2 C_{m_{ac}} dy$$
 (4.34)

Thus, if the wing is straight and has constant airfoil cross section (i.e.  $C_{m_{ac}}$  is constant across the wing span), then we have  $\bar{c} = c$ . However, if c is not constant (e.g in a tapered wing) and we assume that  $C_{M_{ac}} = C_{m_{ac}}$  and  $C_l$  are constant across the wing span, then the mean aerodynamic chord  $\bar{c}$  is simply,

$$\bar{c} = \frac{1}{S} \int_{-b/2}^{b/2} c^2 dy \tag{4.35}$$



(a) Stabilizer-Elevator Configuration



(b) Stabilator Configuration

Figure 4.6: Horizontal Tail Configurations

This integral definition of  $\bar{c}$  is used for any planform. As an example, for a linear tapered wing, we have

$$\bar{c} = \frac{2c_o}{3} \frac{1 + \lambda + \lambda^2}{1 + \lambda} \tag{4.36}$$

where  $c_o$  is the midspan chord (Figure 4.5).

### 4.3 Stick-Free Stability

Wehaveseenintheprevioussectionthekeyelementsinstaticstabilityanalysisforastick-fixedconfiguration. It was assumed that the position of the tail or elevator surface has been fixed by the pilot holding onto the control stick, i.e. to holdthe surfaceintrimmedposition the pilotmust exert aconstant force due to anonzero moment at the elevator hinge. This may not be desireable for long duration flight. Of course, nowadays for highperformanceandlarge-sizeairplane, theproblemisalleviated with theuseofpowerassisted controls and seldom there are unassisted control linkages between the pilot controls and the respective control surfaces.

Nevertheless, it would still be necessary for small-size airplanes to investigate the issue of stick-free stability. It turns out that the effect of freeing the control surface amounts to a reduction in static stability in a certain configuration (e.g. stabilizer-elevator). Let's examine the two basic configurations of horizontal tail surfaces: stabilizer-elevator and stabilator as shown in Figure 4.6.

### **Horizontal Stabilizer-Elevator Configuration**

Let's consider the moment  $H_e$  about the hinge line of the elevator and the corresponding elevator hinge moment coefficient  $C_{h_e}$  defined as,

$$C_{h_e} = \frac{H_e}{1/2\rho V^2 S_e c_e} \tag{4.37}$$

The elevator hinge moment coefficient  $C_{h_e}$  is found to be a function of the tail angle of attack  $\alpha_t$  and of the elevator deflection  $\delta_e$ . As an approximation, one can write

$$C_{h_e} = \frac{\partial C_{h_e}}{\partial \alpha_t} \alpha_t + \frac{\partial C_{h_e}}{\partial \delta_e} \delta_e \tag{4.38}$$

where  $\partial C_{h_e}/\partial \alpha_t$  and  $\partial C_{h_e}/\partial \delta_e$  are assumed constant and determined empirically (i.e. they vary with the configuration of the plan form of the stabilizer-elevator). With the convention that apositive elevator deflection is down, these derivative coefficients are usually negative thus producing a negative hinge moment for any positive change in either  $\alpha_t$  or  $\delta_e$ .

Clearly, the free elevator will reach an equilibrium position when its hinge moment is zero for any tail angle of attack  $\alpha_t$ . Let's denote this angle as  $\delta_{e_{free}}$  which is determined by setting  $C_{h_e}$  equal to zero,

$$C_{h_e} = 0 = \frac{\partial C_{h_e}}{\partial \alpha_t} \alpha_t + \frac{\partial C_{h_e}}{\partial \delta_e} \delta_{e_{free}}$$
(4.39)

This equation allows us to solve for  $\delta_{e_{free}}$  in terms of the angle of attack at the tail  $\alpha_t$ . The tail lift coefficient derived from equation (4.6) is then modified to include the effect of a free elevator as follows,

$$C_{L_t} = a_t \alpha_t + \frac{\partial C_{L_t}}{\partial \delta_e} \delta_e \tag{4.40}$$

However, since for a stick-free case,  $\delta_e = \delta_{e_{free}}$ , equation (4.40) becomes

$$C_{L_t} = a_t \alpha_t - \frac{\partial C_{L_t}}{\partial \delta_e} \frac{\frac{\partial C_{h_e}}{\partial \alpha_t}}{\frac{\partial C_{h_e}}{\partial \delta_e}} \alpha_t$$
(4.41)

or

$$C_{L_t} = F_e a_t \alpha_t \tag{4.42}$$

where

$$F_{e} = 1 - \frac{1}{a_{t}} \frac{\partial C_{L_{t}}}{\partial \delta_{e}} \frac{\frac{\partial C_{h_{e}}}{\partial \alpha_{t}}}{\frac{\partial C_{h_{e}}}{\partial \delta_{e}}} = 1 - \tau \frac{\frac{\partial C_{h_{e}}}{\partial \alpha_{t}}}{\frac{\partial C_{h_{e}}}{\partial \delta_{e}}}$$
(4.43)

where  $\tau = \frac{\partial \alpha_t}{\partial \delta_e}$  is the elevator effectiveness (see Figure 5-33 on page 250 of Perkins & Hage) in

$$\frac{\partial C_{L,t}}{\partial \delta_{e}} = \frac{\partial C_{L,t}}{\partial \alpha_{t}} \frac{\partial \alpha_{t}}{\partial \delta_{e}} = \tau a_{t} \tag{4.44}$$

and  $a_t$  is the lift-curve slope of the tail. The variable  $F_e$  is called the free elevator factor, and it is usually less than unity. Stability analysis for the stick-free case proceeds exactly as in the stick-fixed case. The results are obtained simply by substituting  $a_t$  in equation (4.17) by  $F_e a_t$ . Namely,

$$C_{L_{\alpha}free} = a_w + \eta_t \frac{S_t}{S} F_e a_t (1 - \epsilon_{\alpha}) < C_{L_{\alpha}fixed}$$
(4.45)

The lift curve slope for a stick-free case is always less than that of a stick-fixed case. From the above result for  $C_{L_{\alpha} free}$ , the neutral point  $h_n$  is given by

$$h_{n_{free}} = \frac{h_{ac,w} + \left[\frac{C_{L_{\alpha}free}}{a_{w}} - 1\right]h_{ac,t}}{\frac{C_{L_{\alpha}free}}{a_{w}}} - \left(\frac{C_{M_{\alpha}}}{C_{L_{\alpha}}}\right)_{fus}$$
(4.46)

or

$$h_{n_{free}} = h_{ac,t} - \frac{a_w}{C_{L_{\alpha}free}} (h_{ac,t} - h_{ac,w}) - \left(\frac{C_{M_{\alpha}}}{C_{L_{\alpha}}}\right)_{fus}$$
(4.47)

Notice that since  $C_{L_{\alpha} free} < C_{L_{\alpha} fixed}$ , we deduce that the neutral point for a stick-free case is *ahead* of the neutral point of a stick-fixed case (i.e.  $h_{n_{free}} < h_{n_{fixed}}$ ); hence the stick-free case is less *statically stable* than the stick-fixed case for a given center of gravity position.

From the above, it can be easily shown that

$$C_{M_{\alpha} free} = C_{L_{\alpha} free} \left( h - h_{n_{free}} \right) \tag{4.48}$$

### Example 2

Using results from **Example 1** and assuming that  $C_{h_e} = -0.31\alpha_t - 0.68\delta_e$  (where  $\alpha_t$  and  $\delta_e$  are in radians) with an elevator control effectiveness  $\frac{\partial C_{L_l}}{\partial \delta_e}$  of 1.616 per radians. Then

$$F_{e} = 1 - \frac{1}{a_{t}} \frac{\partial C_{L_{t}}}{\partial \delta_{e}} \frac{\partial C_{h_{e}}/\partial \alpha_{t}}{\partial C_{h_{e}}/\partial \delta_{e}}$$

$$= 1 - \frac{1}{0.0642 \text{per deg}} \frac{1.616 \text{per rad}}{57.3 \text{deg}/\text{rad}} \frac{(-0.31)}{(-0.68)} = 0.80$$
(4.49)

The lift curve slope  $C_{L_{\alpha}}$  is given by

$$C_{L_{\alpha} free} = a_w + \eta_t \frac{S_t}{S} F_e a_t (1 - \epsilon_{\alpha})$$

$$= 0.0731 \text{per deg} + 1 \times \frac{24.4}{160} 0.80 \times 0.0642 \text{per deg} (1 - 0.447)$$

$$= 0.0774 \text{per deg} < C_{L_{\alpha} fixed} = 0.0785 \text{per deg}$$
(4.50)

The neutral point is at

$$h_{n_{free}} = 2.78 - \frac{0.0731 \text{per deg}}{0.0774 \text{per deg}} (2.78 - 0.27) = 0.4094 < h_{n_{fixed}} = 0.443$$
 (4.51)

Then  $C_{M_{\alpha} free} = 0.0744(h - 0.4094)$ .

### **Horizontal Stabilator Configuration**

With this configuration, the elevator deflection is mechanically linked to the horizontal stabilator deflection as follows,

$$\delta_e = k_e i_t + \delta_o \tag{4.52}$$

The deflection  $\delta_o$  is used to provide zero stick force at trim. The hinge moment at the horizontal tail is given by

$$C_{h_t} = \frac{\partial C_{h_t}}{\partial \alpha_t} \alpha_t + \frac{\partial C_{h_t}}{\partial \delta_e} \delta_e \tag{4.53}$$

Recall that the tail angle of attack  $\alpha_t$  in a wing-tail configuration is given by  $\alpha_t = i_t + \alpha_w - \epsilon$  (as in equation (4.6)). Thus the *floating* incidence angle  $i_t$  at the horizontal stabilator is obtained by letting  $C_{h_t} = 0$ ,

$$C_{h_t} = 0 = \frac{\partial C_{h_t}}{\partial \alpha_t} (i_t + \alpha_w - \epsilon) + \frac{\partial C_{h_t}}{\partial \delta_e} (k_e i_t + \delta_o)$$
(4.54)

or

$$i_t = B_e \{ \frac{\partial C_{h_t}}{\partial \alpha_t} (1 - \epsilon_\alpha) \alpha_w + \frac{\partial C_{h_t}}{\partial \delta_e} \delta_o \}$$
 (4.55)

where the constant  $B_e$  is defined as

$$B_e = \frac{-1}{\frac{\partial C_{h_t}}{\partial \alpha_e} + \frac{\partial C_{h_t}}{\partial \delta_e} k_e} \tag{4.56}$$

With the above tail incidence angle expressed as a function of  $\alpha_w$  and  $\delta_o$ , one can then determine the corresponding tail lift coefficient as follows,

$$C_{L_t} = a_t \alpha_t + \frac{\partial C_{L_t}}{\partial \delta_e} \delta_e \tag{4.57}$$

After some simple algebra that proceeds roughly along the following line,

$$C_{L_t} = a_t(i_t + \alpha_w - \epsilon) + \frac{\partial C_{L_t}}{\partial \delta_e} (k_e i_t + \delta_o)$$
(4.58)

$$C_{L_t} = F_e a_t (1 - \epsilon_\alpha) \alpha_w + G_e \delta_o \tag{4.59}$$

where

$$F_e = 1 + (1 + \frac{1}{a_t} \frac{\partial C_{L_t}}{\partial \delta_e} k_e) B_e \frac{\partial C_{h_t}}{\partial \alpha_t}$$
(4.60)

and

$$G_e = (a_t + \frac{\partial C_{L_t}}{\partial \delta_e} k_e) B_e \frac{\partial C_{h_t}}{\partial \alpha_t} + \frac{\partial C_{L_t}}{\partial \delta_e}$$
(4.61)

Note that in this case, the free elevator factor  $F_e$  can be greater than unity; hence resulting in an improvement on static margin for the stabilator configuration.

#### **Example 3:[Anderson]**

For a wing-body combination, the aerodynamic center lies 0.05c ahead of the center of gravity. The moment coefficient about the aerodynamic center is  $C_{M_{ac,wb}} = -0.016$ . If the lift coefficient is  $C_{L_{wb}} = 0.45$ , what is the moment coefficient about the center of gravity?

Note that  $C_{M_{cg,wb}} = C_{M_{ac,wb}} + C_{L_{wb}}(h - h_{ac,wb})$  where  $h - h_{ac,wb} = 0.05$ ,  $C_{L_{wb}} = 0.45$  and  $C_{M_{ac,wb}} = -0.016$ . Thus,  $C_{M_{cg,wb}} = -0.016 + 0.45(0.05) = 0.0065$ .

### **Example 4:[Anderson]**

A wing-body model is tested in a subsonic wind tunnel. The lift is found to be zero at a geometric angle of attack  $\alpha = -1.5^{\circ}$ . At  $\alpha = 5^{\circ}$ , the lift coefficient is measured as 0.52. Also at  $\alpha = 1.0^{\circ}$  and 7.88°, the moment coefficients about the center of gravity are measured as -0.01 and 0.05, respectively. The center of gravity is located at hc = 0.35c. Determine the location of the aerodynamic center and the moment coefficient about the aerodynamic center  $C_{M_{ac.vnb}}$ .

Knowing the lift coefficients at different angles of attack ( $C_{L_{wb}} = 0$  at  $\alpha = -1.5^o$  and  $C_{L_{wb}} = 0.52$  at  $\alpha = 5^o$ ) one can deduce the lift curve slope (as a linear approximation)  $a_{wb}$  as follows,

$$a_{wb} = \frac{\partial C_{L_{wb}}}{\partial \alpha} = \frac{0.52 - 0}{5 - (-1.5)} = 0.08 \text{per deg}$$
 (4.62)

Measuring the moment coefficients about the center of gravity at two different angles of attack and at the same time we know from previous calculation the lift curve slope, one obtains from

$$C_{M_{ce,wb}} = C_{M_{ac,wb}} + a_{wb}\alpha_{wb}(h - h_{ac,wb})$$
(4.63)

the following two linear equations in two unknowns  $C_{M_{ac,wb}}$  and  $(h - h_{ac,wb})$ ,

$$\begin{array}{rcl}
-0.01 & = & C_{M_{ac,wb}} + 0.08(1+1.5)(h - h_{ac,wb}) \\
0.05 & = & C_{M_{ac,wb}} + 0.08(7.88+1.5)(h - h_{ac,wb})
\end{array} \tag{4.64}$$

From equations (4.64), we can solve for  $C_{M_{ac,wb}}$  and  $(h - h_{ac,wb})$  as

$$C_{M_{ac,wb}} = -0.032, \quad (h - h_{ac,wb}) = 0.11$$
 (4.65)

Since h = 0.35, then  $h_{ac,wb} = 0.35 - 0.11 = 0.24$ .

### **Example 5:[Anderson]**

Consider the wing-body model in **Example 4** above. The area and chord of the wing are  $S=0.1m^2$  and c=0.1m, respectively. Now assume that a horizontal tail is added to the model. The distance of the airplane center of gravity to the tail's aerodynamic center is  $l_t=0.17m$ , the tail area is  $S_t=0.02m^2$ , the tail-setting angle is  $i_t=-2.7^o$ , the tail lift slope is  $a_t=0.1$  per degrees, and from experimental measurement  $\epsilon_o=0^o$  and  $\frac{\partial \epsilon}{\partial \alpha}=\epsilon_\alpha=0.35$ . If  $\alpha=7.88^o$ , what is the moment coefficient  $C_{M_{cg}}$  for this airplane model? Does this airplane have longitudinal static stability and balance? Find the neutral point.

From equation (4.11), we have

$$C_{M_{cg}} = C_{M_{ac,w}} + \frac{q_t S_t c_t}{qSc} C_{M_{ac,t}} - \eta_t V_H a_t (i_t - \epsilon_o) + \{ (h - h_{ac,w}) a_w - \eta_t V_H a_t (1 - \epsilon_\alpha) \} \alpha_w$$
(4.66)

We further assume that the tail has a symmetric airfoil shape where  $C_{M_{ac,t}}=0$ . From previous example, we have  $C_{M_{ac,wb}}=-0.032$ ,  $a_w=0.08$ ,  $\alpha_w=7.88^o+1.5^o=9.38^o$  and  $(h-h_{ac,w})=0.11$ . Furthermore

$$\eta_t = 1 \text{ (assumed )}$$

$$V_H = \frac{S_t l_t}{SC} = \frac{0.02(0.17)}{0.1(0.1)} = 0.34$$
(4.67)

Thus

$$C_{M_{cg}} = -0.032 - 1(0.34)(0.1)(-2.7 - 0) + \{0.11(0.08) - 1(0.34)(0.1)(1 - 0.35)\}9.38 = -0.065 \quad (4.68)$$

For longitudinal static stability, we examine  $C_{M_{\alpha}}$  as given in equation (4.12),

$$C_{M_{\alpha}} = (h - h_{ac,w})a_w - \eta_t V_H a_t (1 - \epsilon_{\alpha}) \tag{4.69}$$

or

$$C_{M_{\alpha}} = 0.11(0.08) - 1(0.34)(0.1)(1 - 0.35)$$
  
= -0.0133 < 0(Statically stable) (4.70)

Is the model longitudinally balanced? To find out we need to determine  $C_{M_o}$  (defined in equation (4.12) and from which we derive the equilibrium angle of attack.

$$C_{M_o} = C_{M_{ac,w}} + \frac{q_t S_t c_t}{q S_c} C_{M_{ac,t}} - \eta_t V_H a_t (i_t - \epsilon_o)$$
(4.71)

or

$$C_{M_o} = -0.032 - 1(0.34)(0.1)(-2.7 - 0) = 0.0598 (4.72)$$

Thus, the equilibrium angle of attack is obtained by letting  $C_{M_{cg}} = 0$  in equation (4.11), or

$$C_{M_{cg}} = C_{M_o} + C_{M_\alpha} \alpha_{equilibrium}$$

$$\Rightarrow \alpha_{equilibrium} = -C_{M_o} / C_{M_\alpha} = -(0.0598)/(-0.0133) = 4.4962^o$$
(4.73)

This angle of attack is within reasonable limits; hence the airplane can be balanced and at the same time it is also statically stable.

The neutral point is given by equation (4.19) as

$$h_n = \frac{h_{ac,wb} + \left[\frac{C_{L_{\alpha}}}{a_{wb}} - 1\right]h_{ac,t}}{\frac{C_{L_{\alpha}}}{a_{wb}}}$$
(4.74)

where

$$C_{L_{\alpha}} = a_{wb} + \eta_t \frac{S_t}{S} a_t (1 - \epsilon_{\alpha})$$

$$= 0.08 + 1(0.02)/(0.1)(0.1)(1 - 0.35) = 0.093$$

$$h_{ac,t} = h + l_t/c = 0.35 + 0.17/0.1 = 2.05$$
(4.75)

Thus,

$$h_n = \frac{0.24 + \left[\frac{0.093}{0.08} - 1\right]2.05}{\frac{0.093}{0.08}} = 0.493 \tag{4.76}$$

One can verifies the above result using equation (4.20), namely

$$h - h_n = C_{M_{\alpha}}/C_{L_{\alpha}}$$

$$0.35 - 0.493 = (-0.0133)/(0.093)$$

$$-0.143 = -0.143$$
(4.77)

In the following we discuss some other effects that enter into our analysis of the longitudinal static stability.

### 4.4 Other Influences on the Longitudinal Stability

### 4.4.1 Influence of Wing Flaps

Changes in the wing flaps affect both trim and stability. Themain aerodynamic effects due to flap deflections are:

- Lowering the flaps has the same effect on  $C_{M_{o,wb}}$  as an increase in wing camber. That is producing a negative increment in  $\Delta C_{M_{o,wb}}$ .
- Theangleof wing-bodyzero-liftischanged tobemorenegative. Since the tailincidence  $i_t$  is measured relative to the wing-body zero lift line, this in effect places a positive increment in the tail incidence angle  $i_t$ .
- Change in the spanwise lift distribution at the wing leads to an increase in downwash at the tail, i.e.  $\epsilon_o$  and  $\frac{\partial \epsilon}{\partial \alpha}$  may increase.

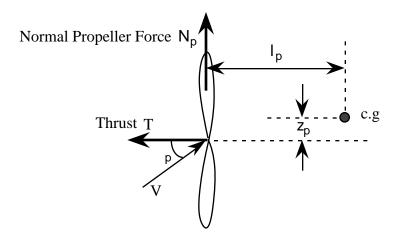


Figure 4.7: Forces on a Propeller

### 4.4.2 Influence of the Propulsive System

The incremental pitching moment about the airplane center of gravity due to the propulsion system (Figure 4.7) is

$$\Delta M_{cg} = Tz_p + N_p l_p \tag{4.78}$$

where T is the thrust and  $N_p$  is the propeller or inlet normal force due to turning of the air. Another influence comes from the increase in flow velocity induced by the propeller or the jet slipstream upon the tail, wing and aft fuselage.

In terms of moment coefficient,

$$\Delta C_{M_{cg}} = \frac{T}{qS} \frac{z_p}{c} + \frac{N_p}{qS} \frac{l_p}{c} \tag{4.79}$$

Since the thrust is directed along the propeller axis and rotates with the airplane, its contribution to the moment about the center of gravity is independent of  $\alpha_w$ . Then we have

$$\Delta C_{M_o} = \frac{T}{qS} \frac{z_p}{c} \tag{4.80}$$

and

$$\Delta C_{M_{\alpha}} = N_{prop} \frac{S_{prop} l_p}{Sc} \frac{\partial C_{N_p}}{\partial \alpha} (1 - \epsilon_{\alpha})$$
(4.81)

where the propeller normal force coefficient  $\partial C_{N_p}/\partial \alpha$  and the downwash (or upwash)  $\epsilon_\alpha$  are usually determined empirically (Figure 4.8).  $N_{prop}$  is the number of propellers and  $S_{prop}$  is the propeller disk area  $(=\pi D^2/4)$  and D is the diameter of the propeller. Notethat a propeller mountedaft of the c.g. isstabilizing. This is one of the advantages of the pusher-propeller configuration. Note that n in Figure 4.8 is the propeller angular speed in rps.

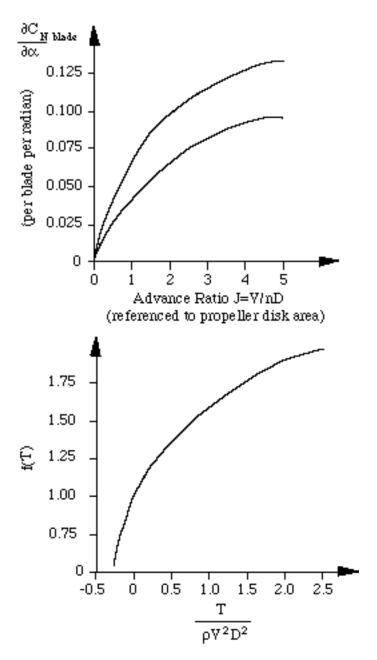


Figure 4.8: Propeller Normal Force Coefficient  $C_{N_{p\alpha}} = \frac{\partial C_{N_{blade}}}{\partial \alpha} f(T)$ 

### 4.4.3 Influence of Fuselage and Nacelles

The pitching moment contributions of the fuselage and nacelles can be approximated as follows (Perkins & Hage p. 229, Equation (5.31)),

$$C_{M_{\alpha fuselage}} = \frac{K_f W_f^2 L_f}{Sc} \text{(per degrees)}$$
 (4.82)

where  $W_f$  is the maximum width of the fuselage or nacelle and  $L_f$  is the length. The empirical pitching moment factor  $K_f$  is given in Figure 4.9 (NACA TR 711).

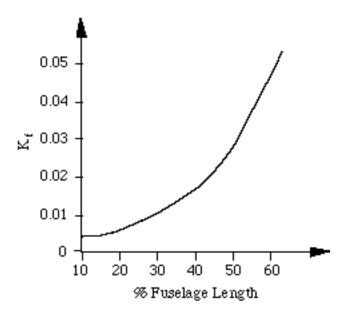


Figure 4.9:  $K_f$  as a Function of the Position of the Wing c/4 Root Chord

### 4.4.4 Effect of Airplane Flexibility

Flexibility of an airframe under aerodynamic loads is evident in any flight vehicle. The phenomenon that couples aerodynamics with structural deformations is studied under the subject of aeroelasticity. There are two types of analysis:

- <u>Static behaviour:</u> Herethesteady-statedeformationsof thevehicle structureareinvestigated. Phenomena such as aileron reversal, wing divergence and reduction in static longitudinal stability fall under this category.
- Dynamic behaviour: The major problem of interest is associated with the phenomena of dynamic loading, buffeting and flutter.

Let's study, for example, the effect of fuselage bending on the tail effectiveness. It can be seen that the angle of attack at the tail is reduced by the fuselage bending according to the following equation,

$$\alpha_t = \alpha_{wb} + i_t - \epsilon - kL_t \tag{4.83}$$

The tail lift coefficient (with  $\delta_e = 0$ ) is

$$C_{L_t} = a_t \alpha_t = a_t (\alpha_{wb} + i_t - \epsilon - kL_t)$$
(4.84)

or

$$C_{L_t} = a_t(\alpha_{wb} + i_t - \epsilon - kq_t S_t C_{L_t})$$

$$\tag{4.85}$$

Solving for  $C_{L_t}$ , we get

$$C_{L_t} = \frac{a_t}{1 + k \eta_t q S_t a_t} (\alpha_{wb} + i_t - \epsilon) \tag{4.86}$$

Thus the tail effectiveness is reduced by a factor  $1/[1 + k\eta_t qS_t a_t]$  that decreases with increasing speed V in the dynamic pressure q. This decrease in the tail lift curve slope will cause the neutral point to move forward (i.e. .reduced static stability).

Similarly, it can be shown that the elevator effectiveness is decreased due to fuselage bending since

$$C_{L_t} = a_t(\alpha_{wb} + i_t - \epsilon - kL_t) + \frac{\partial C_{L_t}}{\partial \delta_e} \delta_e$$
 (4.87)

or solving for  $C_{L_t}$ , we obtain

$$C_{L_t} = \frac{a_t(\alpha_{wb} + i_t - \epsilon) + \frac{\partial C_{L_t}}{\partial \delta_e} \delta_e}{1 + k \eta_t q S_t a_t}$$
(4.88)

Thus the elevator effectiveness is reduced by the same factor  $1/[1 + k\eta_t qS_t a_t]$ .

#### 4.4.5 Influence of Ground Effect

When the airplane is near the ground to within 20% of the wing span, the wing and tail lift curve slope will increase by about 10%. At the same time, the downwash is reduced to about half of the normal value, which requires a greater elevator deflection to hold the nose up. However, static stability is usually improved by the ground effect.

The aircraft must have sufficient elevator effectiveness to trim in ground effect with full flaps and full forward c.g. location, at both power off and full power.

## Chapter 5

## **Static Longitudinal Control**

We have studied in Section 4 the concept of longitudinal stability of an airplane in trim. It was shown that static stability is primarily governed by the sign of the derivative of the moment coefficient about the airplane center of gravity with respect to the angle of attack, i.e  $C_{M_{\alpha}}$ , being *negative*. All the above analysis relies on the fact that one can trim the airplane. The question is what are the controls that allow us to trim the airplane.

### 5.1 Longitudinal Trim Conditions with Elevator Control

For a steady level flight it is easily seen that the airplane velocity in trim is given by

$$V_{trim} = \sqrt{\frac{2W}{\rho SC_{L_{trim}}}} \tag{5.1}$$

Thus if the pilot wants to fly at a lower velocity  $V < V_{trim}$ , then from equation (5.1), we must have  $C_{L_{trim}}$  (or the angle of attack) increased in order to offset the decrease in dynamic pressure. But increasing the angle of attack away from trim would generate for a statically stable airplane a negative pitching moment that tends to bring the angle of attack back to the original trim point (Figure 4.4). It would therefore be impossible to change speed if nothing else is changed about the airplane. It turns out that there are basically two ways to achieve a change in the trim angle of attack. The control concepts are illustrated in Figure 5.1. One possibility is to change the slope of the moment coefficient curve as indicated in Figure 5.1a. Thus, by decreasing the slope  $C_{M_{\alpha}}$  (i.e more negative), one can achieve a smaller trim angle of attack and hence one is able to fly at a faster velocity. If we examine equation (4.12), the only way to modify  $C_{M_{\alpha}}$  is to change the location of the airplane center of gravity that shows up in both the variables h and  $V_H$ . This principle is used extensively in modern hand gliding craft but it is clearly not practical for large fixed wing airplanes. The alternative is tochange the value of  $C_{M_{\alpha}}$  as indicated in Figure 5.1b. It will be shown below that by deflecting the elevator in the horizontal tail one can translate the moment coefficient curve upward and downward while without affecting its slope.

Let's examine the effect of deflecting the elevator on the tail lift coefficient curve. Using a sign convention of positive elevator deflection being downward (or using the right-hand rule for angle), it is clear that a deflected elevator causes the lift curve to shift upward and to the left as shown in Figure 5.2. The lift curve slope remains unchanged. Now, if we assume that the tail is at a constant angle of attack  $\alpha_t$ , say  $\alpha_{t1}$  in Figure 5.2, then an increase in elevator deflection would lead to an increase in tail lift along the vertical dashed line (Figure 5.2). If we plot the tail lift coefficient  $C_{L_t}$  as a function of  $\delta_e$  when the tail is at a given tail angle of

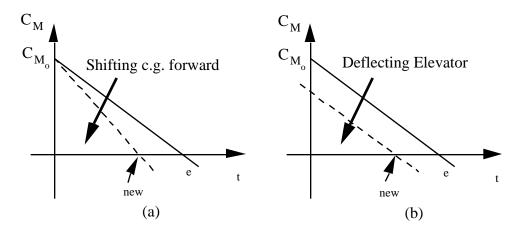


Figure 5.1: How to Change Airplane Trim Angle of Attack

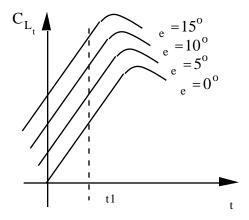


Figure 5.2: Tail Lift Coefficient vs Tail Angle of Attack

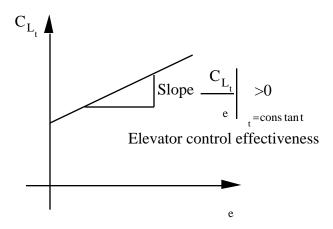


Figure 5.3: Tail Lift Coefficient vs Elevator Deflection

attack  $\alpha_t$  (held constant), we would have a curve much like the one given in Figure 5.3 (where we assume that the slope stays nearly constant and does not change with  $\delta_e$ ). This slope of the tail lift coefficient with respect toelevatordeflectiondenoted by  $\partial C_{L_t}/\partial \delta_e$  is called the *elevatorcontroleffectiveness*. This quantifies the effectiveness of the elevator as a control surface. It can be seen that this constant  $\partial C_{L_t}/\partial \delta_e$  is always *positive*. With the above definition, one can from here on express the tail lift coefficient as a function of two independent variables  $\alpha_t$  and  $\delta_e$ . In the form of a first-order Taylor series expansion, we have

$$C_{L_t} = \frac{\partial C_{L_t}}{\partial \alpha_t} \bigg|_{\delta_e} \alpha_t + \frac{\partial C_{L_t}}{\partial \delta_e} \bigg|_{\alpha_t} \delta_e \tag{5.2}$$

or since  $a_t = \frac{\partial C_{L_t}}{\partial \alpha_t}$ , then

$$C_{L_t} = a_t \alpha_t + \frac{\partial C_{L_t}}{\partial \delta_e} \delta_e \tag{5.3}$$

Substituting equation (5.3) into equation (4.11) for the pitching moment coefficient about the center of gravity, we have

$$C_{M_{cg}} = C_{M_{ac,w}} + \eta_t \frac{S_t c_t}{S_c} C_{M_{ac,t}} + a_w \alpha_w (h - h_{ac,w}) - \eta_t V_H (a_t \alpha_t + \frac{\partial C_{L_t}}{\partial \delta_e} \delta_e)$$
 (5.4)

Then the rate of change of  $C_{M_{cg}}$  due to elevator *only* is defined as  $\frac{\partial C_{M_{cg}}}{\partial \delta_e}$ . From equation (5.4), we obtain

$$\frac{\partial C_{M_{cg}}}{\partial \delta_e} = -\eta_t V_H \frac{\partial C_{L_t}}{\partial \delta_e} \tag{5.5}$$

Note that since  $\frac{\partial C_{L_t}}{\partial \delta_e}$  is always positive, we deduce that  $\frac{\partial C_{M_{cg}}}{\partial \delta_e}$  is always *negative*. Thus, an incremental change in  $C_{M_{cg}}$  for a given elevator deflection  $\delta_e$  is simply,

$$\Delta C_{M_{cg}} = -\eta_t V_H \frac{\partial C_{L_t}}{\partial \delta_e} \delta_e \tag{5.6}$$

So by deflecting the elevator one can shift the moment coefficient curve downward by an amount  $\Delta C_{M_{cg}}$  given in equation (5.6). This confirms the behaviour depicted in Figure 5.1 b, where elevator control can be used to change the trim point. Moreover, from equation (5.4) we can show that the slope of the moment

coefficient curve with respect to angle of attack is not affected (to first-order approximation) by the elevator deflection. Only the value of  $C_{M_0}$  is modified by elevator deflection. Namely,

$$C_{M_{cg}} = (C_{M_o} + \Delta C_{M_{cg}}) + \frac{\partial C_{M_{cg}}}{\partial \alpha_w} \alpha_w$$
  
=  $(C_{M_o} - \eta_t V_H \frac{\partial C_{L_t}}{\partial \delta_e} \delta_e) + \frac{\partial C_{M_{cg}}}{\partial \alpha_w} \alpha_w$  (5.7)

### 5.1.1 Determination of Elevator Angle for a New Trim Angle of Attack

The problem is to find the elevator deflection  $\delta_{etrim}$  such that the moment coefficient equation is balanced at a new angle of attack  $\alpha_n$ . We return to equation (5.7) where we substitute  $\alpha_w$  by  $\alpha_n$  and using equation (5.6) for  $\Delta C_{M_{CR}}$ 

$$C_{M_{cg}} = 0 = C_{M_o} - \eta_t V_H \frac{\partial C_{L_t}}{\partial \delta_e} \delta_{etrim} + \frac{\partial C_{M_{cg}}}{\partial \alpha_w} \alpha_n$$
 (5.8)

Solving for  $\delta_{etrim}$ , we obtain

$$\delta_{etrim} = \frac{C_{M_o} + \frac{\partial C_{M_{cg}}}{\partial \alpha_w} \alpha_n}{\eta_t V_H \frac{\partial C_{L_t}}{\partial \delta_{-}}}$$
(5.9)

### Example 6 [Anderson]

Considerafull-sizeairplanewiththeaerodynamic characteristics defined for the airplane model in **Examples 4 and 5** of Section 4.3. The full-size airplane wing area is  $S = 19m^2$  with a weight of  $W = 2.27 \times 10^4 N$ , and an elevator control effectiveness of 0.04. Determine the elevator deflection angle needed to trim the airplane at a velocity of V = 61m/s at sea level.

First, we need to find the airplane angle of attack to fly at V = 61m/s. It is given by

$$C_L = \frac{2W}{\rho V^2 S} = \frac{2(2.27 \times 10^4)}{1.225(61)^2 (19)} = 0.52$$
 (5.10)

From **Example 5**, we have  $C_{L_{\alpha}} = 0.093$ . Then the absolute angle of attack of the airplane is

$$\alpha_n = \frac{C_L}{C_{L_\alpha}} = \frac{0.52}{0.093} = 5.59^o \tag{5.11}$$

From equation (5.9), we have

$$\delta_{etrim} = \frac{C_{M_o} + \frac{\partial C_{M_{cg}}}{\partial \alpha_w} \alpha_n}{\eta_t V_H \frac{\partial C_{L_t}}{\partial \delta_o}}$$
(5.12)

or

$$\delta_{etrim} = \frac{0.0598 + (-0.0133)(5.59)}{1(0.34)0.04} = -1.0696^{\circ}$$
 (5.13)

#### 5.1.2 Longitudinal Control Position as a Function of Lift Coefficient

From equation (4.16) we have expressed the total lift coefficient as a function of angle of attack at the wing, constant component of downwash and the tail incidence angle,

$$C_L = C_{L_\alpha} \alpha_w + \eta_t \frac{S_t}{S} a_t (i_t - \epsilon_o)$$
 (5.14)

or

$$C_L = C_{L_\alpha} \alpha_w + C_{L_{tit}} (i_t - \epsilon_o) \tag{5.15}$$

with

$$C_{L_{ti_t}} = \eta_t \frac{S_t}{S} a_t \tag{5.16}$$

Note that  $C_{L_{\alpha}}$  is given in equation (4.17). From equation (5.15), we see that  $C_L$  is a linear function of angle of attack  $\alpha_w$ . Thus for a given  $C_L$  and tail incidence angle  $i_t$ , one can solve for  $\alpha_w$  as

$$\alpha_w = \frac{C_L - C_{L_{ti_t}}(i_t - \epsilon_o)}{C_{L_{co}}} \tag{5.17}$$

The other equation of importance is the one for the pitching moment about the airplane center of gravity as given in equations (4.11)-(4.12),

$$C_{M_{cg}} = C_{M_o} + C_{M_\alpha} \alpha_w \tag{5.18}$$

where  $C_{M_o}$  and  $C_{M_\alpha}$  are as defined previously in equations (4.12) where  $C_{M_\alpha}$  can also be expressed as  $C_{M_\alpha} = C_{L_\alpha} (h - h_n)$ . Substituting equation (5.17) and equation (4.12) into equation (5.18) the pitching moment equation is now a function of the total lift coefficient  $C_L$  and the tail incidence angle  $i_t$ . Namely,

$$C_{M_{cg}} = C_{M_{ac,w}} + \eta_t \frac{S_t c_t}{S_c} C_{M_{ac,t}} - \eta_t V_H a_t (i_t - \epsilon_o) + C_{M_\alpha} \frac{C_L - C_{L_{iit}} (i_t - \epsilon_o)}{C_{L_\alpha}}$$
(5.19)

For an airplane in trim, one must have  $C_{M_{cg}} = 0$  in equation (5.19). Then one can solve for the tail incidence angle at a particular  $C_L$  trim,

$$i_t = A_t + B_t C_L \tag{5.20}$$

where

$$A_{t} = \epsilon_{o} + \frac{C_{Mac,w} + \eta_{t} \frac{S_{t}c_{t}}{Sc} C_{Mac,t}}{\frac{C_{Mac}C_{L_{it}}}{C_{La}} + \eta_{t}V_{H}a_{t}}$$

$$= \epsilon_{o} + \frac{C_{Mac,w} + \eta_{t} \frac{S_{t}c_{t}}{Sc} C_{Mac,t}}{\eta_{t} \frac{S_{t}}{S} a_{t} [\frac{lt}{c} + h - h_{n}]}$$
(5.21)

and

$$B_{t} = \frac{C_{M_{\alpha}}}{C_{M_{\alpha}}C_{L_{t_{i}}} + \eta_{t}V_{H}a_{t}C_{L_{\alpha}}}$$

$$= \frac{h - h_{n}}{\eta_{t} \frac{S_{i}}{S} a_{t} \left[\frac{l_{t}}{c} + h - h_{n}\right]}$$
(5.22)

Let's study the sign of the coefficient  $B_t$ . Note that for a statically stable airplane,  $C_{M_{\alpha}} < 0$ . It can be easily shown that the denominator term  $C_{M_{\alpha}}C_{L_{ii_t}} + \eta_t V_H a_t C_{L_{\alpha}}$  is equal to

$$C_{M_{\alpha}}C_{L_{i_{t}}} + \eta_{t}V_{H}a_{t}C_{L_{\alpha}} = \eta_{t}\frac{S_{t}}{S}a_{t}C_{L_{\alpha}}(h - h_{n} + \frac{l_{t}}{C})$$
(5.23)

Thus if  $l_t/c > h_n - h$ , then the coefficient  $B_t$  will be negative. Equation (5.22) allows one to determine experimentally the neutral point. This is done by measuring  $i_t$  as a function of  $C_L$  for differentc.g. locations. The slopes of the experimentally derived curves are then plotted as a function of center of gravity locations (i.e. h). The neutral point  $h_n$  is determined by extrapolation to find the value of c.g. that gives a zero slope in  $\partial i_t/\partial C_L$  (see Figure 5.4). According to equation (5.20) this simply corresponds to having  $C_{M_\alpha} = 0$  or  $h = h_n$  in equation (4.20).

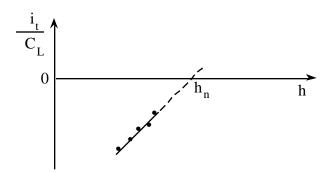


Figure 5.4: Determination of Stick-Fixed Neutral Point from Flight Test

### 5.2 Control Stick Forces

Pilotsusecontrolstickforcesasoneofthemeansofevaluatingtheflyingqualitiesofanairplane. Thus, ability to control an airplane is quantified in terms of required maximum exerted control forces and its sensitivity with respect to airspeed. For simple mechanical control systems, the pilot controls are directly linked to the respective control surfaces and the forces he must exert are proportional to the hinge moment (generated primarily from aerodynamics) about the pivot point at the surfaces.

Let's review the key equations governing the analysis of control hinge moments.

In the design of airplane control system, the stick (or control wheel) forces must lie within acceptable limits throughout the operating envelope (V-n diagram) of the airplane. And the *gradient* of these forces with respect to airspeed at trim point must produce the proper "feel" to the pilot. In general, the pilot tends to push forward in the longitudinal control to fly faster and pull on it to slow down. A requirement stated in FAR Part 23 poses a limit on the maximum force of 60 *lbs* for the stick and 75 *lbs* for the control wheel.

For analysis, we often assume that the control force P is proportional to the hinge moment H at the elevator. Namely,

$$P = GH (5.24)$$

It can be shown that for a system in equilibrium we have  $Ps + H\delta = 0$ . Or

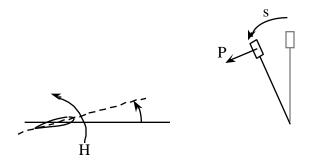


Figure 5.5: Longitudinal Control Stick to Stabilator

$$P = -\frac{\delta}{s}H\tag{5.25}$$

Thus,  $G = -\delta/s$  is the gearing ratio which, as derived here, is totally independent of the details of the mechanical linkage. Since  $\delta$  is negative for a positive stick displacement s as shown in Figure 5.5, the hinge moment would be positive for a positive stick force; thus G is positive.

#### 5.2.1 Stick Force for a Stabilator

For a symmetrical airfoil, the pitching moment at the horizontal stabilator is given by

$$C_{M_t} = C_{M_{tot}} (=0) + C_{M_{tot}} \alpha_t + C_{M_t \delta_s} \delta_e$$
 (5.26)

From equation (5.24), we deduce the stick force for the stabilator to be

$$P = Gq_t S_t c_t \{ C_{M_{tot}} \alpha_t + C_{M_t \delta_e} \delta_e \}$$
(5.27)

Note that  $C_{M_t \alpha_t}$  can be positive or negative depending on whether the aerodynamic center of the stabilator is ahead or behind the pivot. The coefficient  $C_{M_t \delta_e}$  is generally negative. The first step is to express  $\alpha_t$  and  $\delta_e$  in terms of  $C_L$ . First we recall that the elevator deflection is linked to the horizontal tail incidence as

$$\delta_e = k_e i_t + \delta_o \tag{5.28}$$

The trim angle of attack  $\alpha_w$  is determined from the total lift coefficient and the tail angle of incidence  $i_t$  from the pitching moment equation in balance. Namely,

$$C_L = a_w \alpha_w + \eta_t \frac{S_t}{S} \{ a_t [i_t - \epsilon_o + (1 - \epsilon_\alpha) \alpha_w] + \frac{\partial C_{L_t}}{\partial \delta_e} (k_e i_t + \delta_o) \}$$
 (5.29)

or

$$C_L = C_{L_{\alpha}}\alpha_w + C_{L_{it_t}}F_e i_t + \eta_t \frac{S_t}{S} \left[ \frac{\partial C_{L_t}}{\partial \delta_e} \delta_o - a_t \epsilon_o \right]$$
 (5.30)

where  $C_{L_{\alpha}}$  is given in equation (4.17),

$$F_e = 1 + \frac{1}{a_t} \frac{\partial C_{L_t}}{\partial \delta_e} k_e \tag{5.31}$$

and

$$C_{L_{ti_t}} = \eta_t \frac{S_t}{S} a_t \tag{5.32}$$

One can solve for  $\alpha_w$  in terms of  $C_L$ ,  $i_t$ ,  $\delta_o$ ,  $\epsilon_o$  as follows,

$$\alpha_w = \frac{1}{C_{L_a}} \{ C_L - \eta_t \frac{S_t}{S} \frac{\partial C_{L_t}}{\partial \delta_e} \delta_o + \eta_t \frac{S_t}{S} a_t \epsilon_o - C_{L_{ii_t}} F_e i_t \}$$
 (5.33)

The other equation we use is the pitching moment about the airplane center of gravity,

$$C_{M_{cg}} = C_{M_o} + C_{M_\alpha} \alpha_w \tag{5.34}$$

where

$$C_{M_o} = C_{M_{ac,w}} + \eta_t \frac{S_t c_t}{S_c} C_{M_{ac,t}} - \eta_t V_H [a_t (F_e i_t - \epsilon_o) + \frac{\partial C_{L_t}}{\partial \delta_e} \delta_o]$$
 (5.35)

and

$$C_{M_{\alpha}} = (h - h_{ac,w})a_w - \eta_t V_H a_t (1 - \epsilon_{\alpha}) = C_{L_{\alpha}}(h - h_n)$$
 (5.36)

Using equations (5.28), (5.33) and the fact that in trim  $C_{M_{cg}} = 0$ , we can solve for  $i_t$  in terms of  $C_L$ ,  $\delta_o$ ,  $\epsilon_o$ . We obtain

$$i_t = A_s + B_s C_L \tag{5.37}$$

where

$$A_{s} = \frac{1}{\eta_{t} \frac{S_{t}}{S} a_{t} F_{e}(\frac{l_{t}}{c} + h - h_{n})} \{ C_{M_{ac,w}} + \eta_{t} \frac{S_{t} C_{t}}{Sc} C_{M_{ac,t}} \} + \frac{1}{F_{e}} \epsilon_{o} - \frac{1}{a_{t} F_{e}} \frac{\partial C_{L_{t}}}{\partial \delta_{e}} \delta_{o}$$
 (5.38)

and

$$B_{s} = \frac{h - h_{n}}{\eta_{t} \frac{S_{t}}{S} a_{t} F_{e}(\frac{l_{t}}{c} + h - h_{n})}$$
(5.39)

Substituting  $i_t$  of equation (5.37) into equation (5.33), we obtain

$$\alpha_w = A_a + B_a C_L \tag{5.40}$$

where

$$A_{a} = -\frac{1}{(\frac{l_{t}}{c} + h - h_{n})C_{L_{\alpha}}} \{C_{M_{ac,w}} + \eta_{t} \frac{S_{t}c_{t}}{Sc} C_{M_{ac,t}}\}$$
(5.41)

and

$$B_{a} = \frac{\frac{l_{t}}{c}}{(\frac{l_{t}}{c} + h - h_{n})C_{L_{\alpha}}}$$
 (5.42)

The stick force P given in equation (5.27) becomes

$$\frac{P}{Gn_t a S_t c_t} = \left[ (C_{M_t \alpha_t} + C_{M_t \delta_e} k_e) i_t + C_{M_t \alpha_t} (1 - \epsilon_\alpha) \alpha_w - C_{M_t \alpha_t} \epsilon_o + C_{M_t \delta_e} \delta_o \right]$$
(5.43)

or

$$\frac{P}{G\eta_{t}qS_{t}c_{t}} = \left[ (C_{M_{t\alpha_{t}}} + C_{M_{t\delta_{e}}}k_{e})(A_{s} + B_{s}C_{L}) + C_{M_{t\alpha_{t}}}(1 - \epsilon_{\alpha})(A_{a} + B_{a}C_{L}) - C_{M_{t\alpha_{t}}}\epsilon_{o} + C_{M_{t\delta_{e}}}\delta_{o} \right] (5.44)$$

Thus the stick force P is a linear function of  $C_L$ . By collecting all the terms we have

$$\frac{P}{G\eta_t q S_t c_t} = \bar{A} + \bar{B}C_L \tag{5.45}$$

where

$$\bar{A} = [(C_{M_{t \alpha_{t}}} + C_{M_{t \delta_{e}}} k_{e}) A_{s} + C_{M_{t \alpha_{t}}} (1 - \epsilon_{\alpha}) A_{a} - C_{M_{t \alpha_{t}}} \epsilon_{o} + C_{M_{t \delta_{e}}} \delta_{o}]$$
(5.46)

and

$$\bar{B} = [(C_{M_{t \alpha_t}} + C_{M_{t \delta_e}} k_e) B_s + C_{M_{t \alpha_t}} (1 - \epsilon_\alpha) B_a]$$
(5.47)

Notice that the parameter  $\delta_o$  in  $\bar{A}$  can be used to achieve P=0 at a particular trim velocity  $V_{trim}$ . That is,

$$0 = \bar{A} + \bar{B}C_{L_{trim}} \tag{5.48}$$

or  $\bar{A} = -\bar{B}C_{L_{trim}}$  . We can substitute  $C_L$  by

$$C_L = \frac{W}{qS} = \frac{W}{\frac{1}{2}\rho V^2 S}$$
 (5.49)

Using equations (5.45) and (5.49), it can be easily shown that

$$\frac{P}{G\eta_t S_t c_t} = q \bar{B} \left( C_L - C_{L_{trim}} \right)$$

$$= q \bar{B} C_L \left( 1 - \frac{C_{L_{trim}}}{C_L} \right)$$

$$= \left( \frac{W}{S} \right) \bar{B} \left\{ 1 - \frac{V^2}{V_{trim}^2} \right\}$$
(5.50)

Note that  $V_{trim}$  is given in equation (5.1) and W/S is simply the wing loading.

From the above equation, we can obtain the gradient of the stick force with respect to speed at  $V = V_{trim}$ ,

$$\frac{dP}{dV}\Big|_{V=V_{trim}} = -2G\eta_t S_t c_t \frac{W}{S} \frac{\bar{B}}{V_{trim}}$$
(5.51)

For a given trim speed, a proper stick force gradient as stated in FAR Part 23 must be negative for all conditions of flight. Or  $\bar{B}$  must be positive. Notice that the gradient is large if  $V_{trim}$  is small. Figure 5.6 shows a typical stick force versus speed curve described in equation (5.51). At V=0, clearly from equation (5.50) we have

$$P = G\eta_t S_t c_t \frac{W}{S} \bar{B} \tag{5.52}$$

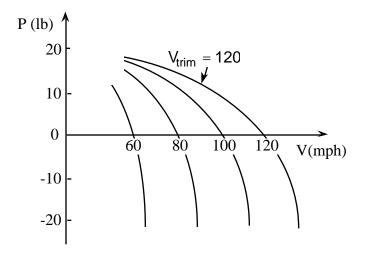


Figure 5.6: Stick Force versus Velocity Curve

### 5.2.2 Stick Force for a Stabilizer-Elevator Configuration

We proceed as before by first giving the hinge moment associated with the stick force P,

$$C_H = C_{H_o} + \frac{\partial C_H}{\partial \alpha_t} \alpha_t + \frac{\partial C_H}{\partial \delta_e} \delta_e + \frac{\partial C_H}{\partial \delta_t} \delta_t$$
 (5.53)

where in general  $C_{H_o} = 0$ . Next we examine the total lift coefficient

$$C_L = C_{L_{\alpha}} \alpha_w + \eta_t \frac{S_t}{S} \frac{\partial C_{L_t}}{\partial \delta_e} \delta_e - \eta_t \frac{S_t}{S} a_t \epsilon_o$$
 (5.54)

where we assume that the lift contributed by the trim tab is negligeable. Solve  $\alpha_w$  in terms of  $\delta_e$  and  $C_L$ ,

$$\alpha_w = \frac{1}{C_{L_\alpha}} \{ C_L - \eta_t \frac{S_t}{S} \frac{\partial C_{L_t}}{\partial \delta_e} \delta_e + \eta_t \frac{S_t}{S} a_t \epsilon_o \}$$
 (5.55)

Now we consider the pitching moment,

$$C_{M_{cg}} = C_{M_o} + C_{M_\alpha} \alpha_w \tag{5.56}$$

where

$$C_{M_o} = C_{M_{ac,w}} + \eta_t \frac{S_t c_t}{S_c} C_{M_{ac,t}} + \eta_t V_H [a_t \epsilon_o - \frac{\partial C_{L_t}}{\partial \delta_e} \delta_e]$$
 (5.57)

and

$$C_{M_{\alpha}} = (h - h_{ac,w})a_w - \eta_t V_H a_t (1 - \epsilon_{\alpha})$$

$$(5.58)$$

From trim balance with  $C_{M_{cg}} = 0$ , we solve for  $\delta_e$ 

$$\delta_e = A_e + B_e C_L \tag{5.59}$$

where

$$A_{e} = \frac{1}{\eta_{t} \frac{S_{t}}{S} \frac{\partial C_{L_{t}}}{\partial \delta_{e}} (\frac{l_{t}}{c} + h - h_{n})} \{ C_{M_{ac,w}} + \eta_{t} \frac{S_{t} c_{t}}{Sc} C_{M_{ac,t}} \} + \frac{a_{t}}{\frac{\partial C_{L_{t}}}{\partial \delta_{e}}} \epsilon_{o}$$

$$(5.60)$$

and

$$B_e = \frac{h - h_n}{\eta_t \frac{S_t}{S} \frac{\partial C_{L_t}}{\partial \delta_n} (\frac{l_t}{c} + h - h_n)}$$
(5.61)

Similarly, by substituting equation (5.59) in equation (5.55), we can express  $\alpha_w$  in terms of  $C_L$ ,

$$\alpha_w = A_a + B_a C_L \tag{5.62}$$

where

$$A_{a} = -\frac{1}{(\frac{l_{t}}{c} + h - h_{n})C_{L_{\alpha}}} \{C_{M_{ac,w}} + \eta_{t} \frac{S_{t}c_{t}}{Sc} C_{M_{ac,t}}\}$$
(5.63)

and

$$B_a = \frac{\frac{l_t}{c}}{(\frac{l_t}{c} + h - h_n)C_{L_\alpha}}$$
 (5.64)

From equations (5.24)) and (5.53), the stick force P becomes

$$\frac{P}{G\eta_t q S_e c_e} = \left[\frac{\partial C_H}{\partial \alpha_t} (1 - \epsilon_\alpha) \alpha_w + \frac{\partial C_H}{\partial \delta_e} \delta_e + \frac{\partial C_H}{\partial \delta_t} \delta_t\right]$$
 (5.65)

Thus the stick force P is a linear function of  $C_L$ . By collecting all the terms we have

$$\frac{P}{G\eta_t q S_e c_e} = \bar{A}_e + \bar{B}_e C_L \tag{5.66}$$

where

$$\bar{A}_e = \left[\frac{\partial C_H}{\partial \alpha_t} (1 - \epsilon_\alpha) A_a + \frac{\partial C_H}{\partial \delta_e} A_e + \frac{\partial C_H}{\partial \delta_t} \delta_t\right]$$
 (5.67)

and

$$\bar{B}_e = \left[\frac{\partial C_H}{\partial \alpha_t} (1 - \epsilon_\alpha) B_a + \frac{\partial C_H}{\partial \delta_e} B_e\right]$$
 (5.68)

Again the trim tab  $\delta_t$  is used to achieve P = 0 at  $V = V_{trim}$ , then one can simplify the above equation for the stick force P to the following,

$$\frac{P}{G\eta_t S_e c_e} = (\frac{W}{S})\bar{B}_e \{1 - \frac{V^2}{V_{trim}^2}\}$$
 (5.69)

From the above equation, we obtain the gradient of the stick force with respect to speed at  $V = V_{trim}$ ,

$$\frac{dP}{dV}\Big|_{V=V_{trim}} = -2G\eta_t S_e c_e \frac{W}{S} \frac{\bar{B}_e}{V_{trim}}$$
(5.70)

### **5.3** Steady Maneuver

We now consider the determination of the elevator angle per g in a pull-up maneuver. In the analysis, the concepts of stick-fixed and stick-free maneuver margins are introduced. For an airplane in a steady pull-up maneuver, the lift force will exceed the vehicle weight. Namely,

$$L = W(1 + \frac{a_n}{g}) \tag{5.71}$$

where  $a_n$  is the vehicle acceleration. The lift to weight ratio is known as the load factor n or

$$n = 1 + \frac{a_n}{g} \tag{5.72}$$

For a straight level flight,  $a_n = 0$  and n = 1.

We examine the pitching moment of the airplane in this pull-up maneuver from which we derive the quantity known as *elevator angle per g*. Again we have

$$C_{M_{cg}} = C_{M_o} + C_{M_\alpha} \alpha_w + \Delta C_M$$
 (due to airplane rotation in a steady pull-up maneuver) (5.73)

where

$$\Delta C_M = -\frac{q \eta_t S_t a_t \Delta \alpha_t l_t}{q S_C} = -\eta_t V_H a_t \Delta \alpha_t \tag{5.74}$$

The incremental angle of attack at the tail due to a constant angular velocity Q is

$$\Delta \alpha_t = \frac{l_t Q}{V} \tag{5.75}$$

or

$$\Delta \alpha_t = \frac{2l_t}{c} \bar{q} \tag{5.76}$$

with  $\bar{q}$  being the dimensionless pitch rate variable defined as

$$\bar{q} = \frac{Qc}{2V} \tag{5.77}$$

### 5.3.1 Horizontal Stabilizer-Elevator Configuration: Elevator per g

When the airplane is in straight and level flight (unaccelerated), the elevator angle and stick force to trim are  $\delta_e$  and P respectively. When in the pull-up maneuver, the elevator is deflected to  $\delta_e + \Delta \delta_e$  and the stick force required is  $P + \Delta P$ . The quantities  $\Delta \delta_e/(a_n/g)$  and  $\Delta P/(a_n/g)$  are known as the *elevator angle per g* and the *stick force per g* respectively. These are measures of airplane maneuverability; the smaller they are the more maneuverable it is.

Including the effects due to pitch rotation Q in a pull-up maneuver, the pitching moment becomes

$$C_{M_{cg}} = C_{M_o} + C_{M_\alpha} \alpha_w + \frac{\partial C_M}{\partial \delta_e} \delta_e + \frac{\partial C_M}{\partial Q} Q$$
 (5.78)

Note that Q is <u>not</u> dimensionless and it has the units of rad/s or deg/s. To nondimensionalize Q we use the variable  $\bar{q}$  defined in equation (5.77). Then equation (5.78) becomes,

$$C_{M_{cg}} = C_{M_o} + C_{M_\alpha} \alpha_w + \frac{\partial C_M}{\partial \delta_e} \delta_e + C_{M_{\bar{q}}} \bar{q}$$
 (5.79)

We can derive  $C_{M_{\bar{q}}}$  from  $\Delta C_M$  and the expression of  $\Delta \alpha_t$  due to Q,

$$C_{M_{\bar{q}}} = -\eta_t a_t V_H \frac{2l_t}{c} \tag{5.80}$$

This term  $C_{M_{\bar{q}}}$  is often referred to as the *pitch damping* term, since it produces a negative pitching moment due to a change in pitch rate.

In a steady pull-up there is still no angular acceleration in the pitch axis, thus for equilibrium  $C_{M_{cg}} = 0$  as it is in trimmed straight level flight condition. The increment  $\Delta C_M$  resulting from a steady maneuver is

$$\Delta C_{M_{cg}} = 0 = C_{M_{\alpha}} \Delta \alpha_w + C_{M_{\delta_e}} \Delta \delta_e + C_{M_{\bar{q}}} \bar{q}$$
(5.81)

where we assume that  $C_{M_o}$  remains constant in the maneuver. From the above equation, one can solve for  $\Delta \delta_e$  as

$$\Delta \delta_e = -\frac{C_{M_\alpha} \Delta \alpha_w + C_{M_{\bar{q}}} \bar{q}}{C_{M_{\delta_e}}} \tag{5.82}$$

It can be shown that since  $a_n = QV$  we have

$$\bar{q} = \frac{Qc}{2V} = \frac{a_n c}{2V^2} = \frac{gc}{2V^2} (\frac{a_n}{g})$$
 (5.83)

Again the incremental change in angle of attack  $\Delta \alpha_w$  is determined from

$$\Delta C_L = C_{L_\alpha} \Delta \alpha_w + \frac{\partial C_L}{\partial \delta_e} \Delta \delta_e \tag{5.84}$$

where  $\Delta C_L$  is the incremental lift coefficient due to the pull-up; namely

$$\frac{C_L + \Delta C_L}{C_L} = \frac{L}{W} = 1 + \frac{a_n}{g} \tag{5.85}$$

or

$$\Delta C_L = \frac{a_n}{g} C_L \text{(for 1 g flight)}$$
 (5.86)

Then

$$\Delta \alpha_w = \frac{C_L}{C_{L_\alpha}} \frac{a_n}{g} - \frac{C_{L_{\delta_e}}}{C_{L_\alpha}} \Delta \delta_e$$
 (5.87)

Note that

$$C_{L_{\delta_e}} = \frac{\partial C_L}{\partial \delta_e} = \eta_t \frac{S_t}{S} \frac{\partial C_{L_t}}{\partial \delta_e}$$
 (5.88)

or

$$C_{L_{\delta_e}} = \eta_t \frac{S_t}{S} \frac{\partial C_{L_t}}{\partial \alpha_t} \frac{\partial \alpha_t}{\partial \delta_e} = \eta_t \frac{S_t}{S} \frac{\partial C_{L_t}}{\partial \alpha_t} \tau = \eta_t \frac{S_t}{S} a_t \tau \tag{5.89}$$

where the coefficient  $\tau$  can be determined from Figure 5.33 (Perkins & Hage p. 250).

Substituting equations (5.83) and (5.87) into equation (5.82) we have

$$\frac{\Delta \delta_e}{(a_n/g)} = -\frac{C_{M_\alpha} C_L + C_{M_{\bar{q}}} C_{L_\alpha} (\frac{gc}{2V^2})}{C_{M_{\delta_e}} C_{L_\alpha} - C_{M_\alpha} C_{L_{\delta_e}}}$$
(5.90)

Note that  $W = qSC_L = \frac{1}{2}\rho V^2SC_L$  and W = mg where m is the mass of the airplane. Hence,  $C_L = \frac{2W}{\rho V^2S}$ . Also recall that  $C_{M_{\delta_e}} = -\eta_t V_H C_{L_t}$  from equation (5.6). Then

$$\frac{\Delta \delta_e}{(a_n/g)} = -C_L C_{L_\alpha} \frac{h - h_n + \frac{\rho Sc}{4m} C_{M_{\bar{q}}}}{C_{M_{\delta_e}} C_{L_\alpha} - C_{M_\alpha} C_{L_{\delta_e}}}$$
(5.91)

or letting  $\mu = \frac{2m}{\alpha SC}$  (known as the relative mass parameter), then

$$\frac{\Delta \delta_e}{(a_n/g)} = -C_L C_{L_\alpha} \frac{h - h_n + \frac{1}{2\mu} C_{M_{\bar{q}}}}{C_{M_{\delta_e}} C_{L_\alpha} - C_{M_\alpha} C_{L_{\delta_e}}}$$

$$(5.92)$$

Similarly to the neutral point, there is also a particular value of h, known as the stick-fixed maneuver point denoted here by  $h_m$ , forwhich no (i.e. very small) elevator willbe required toproduce a finite (i.e. not small) acceleration. This is determined from equation (5.92) by setting  $\Delta \delta_e = 0$ ; namely

$$h_m = h_n - \frac{1}{2\mu} C_{M_{\bar{q}}} \tag{5.93}$$

Since  $C_{M_{\bar{q}}} < 0$ , then  $h_m > h_n$  or the stick-fixed maneuver point lies aft of the neutral point. The quantity  $h_m - h$  is known as the stick-fixed maneuver margin. And we can rewrite equation (5.92) as

$$\frac{\Delta \delta_e}{(a_n/g)} = \frac{C_L C_{L_\alpha}(h_m - h)}{C_{M_{\delta_e}} C_{L_\alpha} - C_{M_\alpha} C_{L_{\delta_e}}}$$
(5.94)

$$\frac{\Delta \delta_e}{(a_n/g)} = \frac{C_L(h_m - h)}{C_{L_{t_{\delta_e}}} \left[ -\eta_t V_H - (h - h_n) \eta_t \frac{S_t}{S} \right]}$$
(5.95)

or

$$\frac{\Delta \delta_e}{(a_n/g)} = \frac{C_L(h_m - h)}{C_{L_{t_h}} \eta_t \frac{S_t}{S} (h_n - h_{ac,t})} < 0$$
 (5.96)

### 5.3.2 Horizontal Stabilator Configuration: Elevator per g

A similar derivation can be performed for the stabilator configuration and we obtain the following results,

$$\frac{\Delta i_t}{(a_n/g)} = -C_L C_{L_\alpha} \frac{h - h_n + \frac{1}{2\mu} C_{M_q}}{C_{M_{i_t}} C_{L_\alpha} - C_{M_\alpha} C_{L_{i_t}}}$$
(5.97)

where we replace  $C_{M_{\delta_{\rho}}}$  by  $C_{M_{i_t}}$  and  $C_{L_{\delta_{\rho}}}$  by  $C_{L_{i_t}}$ .

### 5.3.3 Stabilizer-Elevator Configuration: Stick Force per g

Recall from equation (5.53) that

$$C_H = \frac{\partial C_H}{\partial \alpha_t} \alpha_t + \frac{\partial C_H}{\partial \delta_e} \delta_e + \frac{\partial C_H}{\partial \delta_t} \delta_t$$
 (5.98)

assuming that the trim tab has negligeable contribution to the total lift and moment. In a steady pull-up,

$$\alpha_t = i_t + \alpha_w - \epsilon + 2\frac{l_t}{c}\bar{q} \tag{5.99}$$

or, assuming  $\epsilon_o = 0$ ,

$$\alpha_t = i_t + (1 - \epsilon_\alpha)\alpha_w + 2\frac{l_t}{c}\bar{q} \tag{5.100}$$

Then the stick force P is given by

$$\frac{P}{G\eta_t q S_e c_e} = \left[\frac{\partial C_H}{\partial \alpha_t} \left[i_t + (1 - \epsilon_\alpha)\alpha_w + \frac{2l_t}{c}\bar{q}\right] + \frac{\partial C_H}{\partial \delta_e} \delta_e + \frac{\partial C_H}{\partial \delta_t} \delta_t\right]$$
(5.101)

Let  $\delta_t$  be adjusted to achieve P=0 at straight level flight (unaccelerated). Then  $\alpha_w=\alpha_o+\Delta\alpha_w$ ,  $\delta_e=\delta_{eo}+\Delta\delta_e$ . It then follows that

$$\frac{\Delta P}{G\eta_t q S_e c_e} = \left[\frac{\partial C_H}{\partial \alpha_t} \left[ (1 - \epsilon_\alpha) \Delta \alpha_w + \frac{2l_t}{c} \bar{q} \right] + \frac{\partial C_H}{\partial \delta_e} \Delta \delta_e \right]$$
 (5.102)

Recall that

$$\frac{2l_t}{c}\bar{q} = \frac{gl_t}{V^2}(\frac{a_n}{g})\tag{5.103}$$

The variables  $\Delta \alpha_w$  and  $\Delta \delta_e$  are those obtained in the previous analysis for elevator per gas given inequations (5.87) and (5.96) respectively. Thus, equation (5.102) becomes

$$\frac{\Delta P}{G\eta_t q S_e c_e} = \left[\frac{\partial C_H}{\partial \alpha_t} \left[ (1 - \epsilon_\alpha) \left\{ \frac{C_L}{C_{L_\alpha}} \left( \frac{a_n}{g} \right) - \frac{C_{L_{\delta_e}}}{C_{L_\alpha}} \Delta \delta_e \right\} + \frac{g l_t}{V^2} \left( \frac{a_n}{g} \right) \right] + \frac{\partial C_H}{\partial \delta_e} \Delta \delta_e \right]$$
 (5.104)

After some simple manipulations, we obtain

$$\frac{\Delta P}{(a_n/g)} = G\eta_t S_e c_e(\frac{W}{S}) \left\{ \frac{\partial C_H}{\partial \alpha_t} \left[ \frac{(1 - \epsilon_\alpha)}{C_{L_\alpha}} + \frac{1}{\mu} \frac{l_t}{c} \right] + \frac{\frac{\partial C_H}{\partial \delta_e} C_{L_\alpha} - \frac{\partial C_H}{\partial \alpha_t} (1 - \epsilon_\alpha) C_{L_{\delta_e}}}{C_{M_{\delta_e}} C_{L_\alpha} - C_{M_\alpha} C_{L_{\delta_e}}} (h_m - h) \right\}$$
 (5.105)

Again, there is a position of center of gravity h for which  $\Delta P = 0$ . This is known as the *stick-free* maneuver point denoted by  $h'_m$ . It is called the stick-free maneuver point since it corresponds to the fact that the pilot can let go of his control stick (i.e. stick free). Equation (5.105) can be rewritten as

$$\frac{\Delta P}{(a_n/g)} = G\eta_t S_e c_e(\frac{W}{S}) \{ \frac{\frac{\partial C_H}{\partial \delta_e} C_{L_\alpha} - \frac{\partial C_H}{\partial \alpha_t} (1 - \epsilon_\alpha) C_{L_{\delta_e}}}{C_{M_{\delta_e}} C_{L_\alpha} - C_{M_\alpha} C_{L_{\delta_e}}} (h'_m - h) \}$$
 (5.106)

where

$$h'_{m} = h_{m} + \frac{\partial C_{H}}{\partial \alpha_{t}} \left[ \frac{(1 - \epsilon_{\alpha})}{C_{L_{\alpha}}} + \frac{1}{\mu} \frac{l_{t}}{c} \right] \frac{C_{M_{\delta_{e}}} C_{L_{\alpha}} - C_{M_{\alpha}} C_{L_{\delta_{e}}}}{\frac{\partial C_{H}}{\partial \delta_{c}} C_{L_{\alpha}} - \frac{\partial C_{H}}{\partial \alpha_{c}} (1 - \epsilon_{\alpha}) C_{L_{\delta_{e}}}}$$
(5.107)

One would create a catastrophic situation if we load the airplane such that  $h = h'_m$ . In this case, the pilot would inadvertently generate extremely large inertia loads on the airplane by exerting little or no control force. Such a situation rarely occurs if the vehicle has an adequate static margin since the stick-fixed and stick-free maneuver points are always aft of the neutral point.

### 5.3.4 Stabilator Configuration: Stick Force per g

A similar analysis as the one performed for the stabilizer-elevator configuration can also be done for the stabilator configuration. Details are left to the students. (They can be found in the reference book by B. W. McCormick)

#### Remarks:

- Stick force per g is a linear function of h. For a normal range of c.g. (i.e.  $h < h'_m$ ) locations, the force gradient is positive. That is the pilot must exert a pull (positive) force on the stick to maneuver a pitch up maneuver. The airplane is highly maneuverable if this force gradient is small (i.e. when the c.g. location is near the stick-free maneuver point). In this case, the airplane may even be statically unstable.
- Stick force per g is directly proportional to wing loading (W/S).
- It is independent of  $C_L$  or V apart from Mach and Reynold effects.

## Chapter 6

## **Lateral Static Stability and Control**

The concept of static stability and control we studied for the longitudinal axis can also be applied to the lateral axis. The key motion variables in the lateral axis correspond to sideslip (with sideslip angle  $\beta$ , or side velocity v), roll (with roll rate p) and yaw (with yaw rate r). Primary controls are rudder  $\delta_r$  and ailerons  $\delta_a$ .

### 6.1 Yawing and Rolling Moment Equations

The lateral motion of an airplane is described in terms of two tightly coupled motions: yaw about the *z*-body axis (i.e. directional) and roll about the *x*-body axis (i.e. lateral).

Using Figure 6.1, let's first identify the motion variables, the control effectors and their sign conventions used in the lateral-directional stability analysis.

Note that all along we adhere strictly to the right-hand rule for sign conventions:

- x-body axis pointed forward,
- y-body axis pointed to the right wing.
- z-axis pointed down toward the earth,
- Positive yaw motion (clockwise) (i.e. yaw rate r > 0 with yaw angle  $\psi$ ),
- Positive roll motion (right wing down) (i.e. roll rate p > 0 with roll angle  $\phi$ ),
- Positive sideslip  $\beta = \sin^{-1}(v/V)$  corresponding to a positive v-component in side velocity.
- Aileron control positive with right aileron down ( $\delta_{ar} > 0$  again consistent with the right-hand rule where a positive rotation about the positive y-body axis) and left aileron up ( $\delta_{al} > 0$ ). In general, we define aileron angle  $\delta_a$  to be  $\delta_a = \delta_{ar} + \delta_{al}$ . Notice that this definition of positive aileron will produce a *negative* rolling moment.
- Positive rudder deflection ( $\delta_r > 0$ ) is to the left (again according to the right-hand rule a positive rotation about the positive z-body axis). With this definition, a positive rudder will generate a *negative* yawing moment and a *positive* side force. This is similar to the definition that a positive elevator deflection would pitch the airplane down, i.e. producing a *negative* pitching moment and a *positive* contribution to lift.

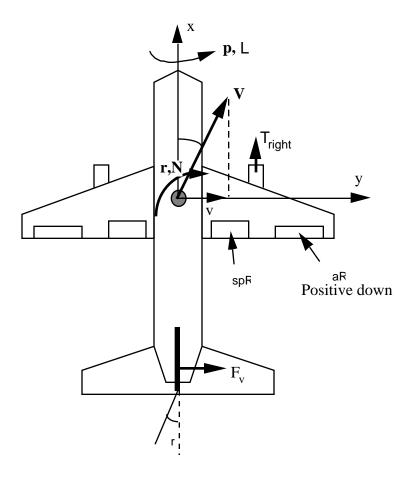


Figure 6.1: Definition of the Lateral Directional Motion of an Airplane

#### **6.1.1** Contributions to the Yawing Moment

There are numerous components that contribute to the yawing moment in the airplane when perturbed in the lateral motion variables. Note that in straight and level flight, we usually have  $\beta = r = p = 0$  in trimmed flight.

The airplane yawing moment about the center of gravity can be written as

$$N_{cg} = N_{wing} + N_{fuselage} + N_{verticaltail} + N_{propulsion} + N_{rudder} + N_{aileron}$$
 (6.1)

or in terms of moment coefficients,

$$C_N = \frac{N}{qSb} = C_{N_{wing}} + C_{N_{fuselage}} + C_{N_{verticaltail}} + C_{N_{propulsion}} + C_{N_{rudder}} + C_{N_{aileron}}$$
(6.2)

- Wing Contribution  $N_{wing}$ : Yawing moment generated at the wing is developed mainly from perturbed motions in sideslip  $\beta$  and roll rate p in the lateral axis. Let's examine some of these effects:
  - Due to sideslip, there is an increase in drag on one side of the wing that is more perpendicular to the flow and thereby would produce a yawing moment. If the wing is swept aft, then this yawing moment is stabilizing (i.e. producing a positive yawing moment to a positive sideslip). An *empirical* formula for the wing yawing moment coefficient due to sideslip,  $C_{N_w,\beta} = \partial C_{N_w}/\partial \beta$ , is given by

$$C_{N_{w,\beta}} = C_L^2 \left\{ \frac{1}{4\pi A} - \left[ \frac{\tan \Lambda}{\pi A(A + 4\cos \Lambda)} \right] \left[ \cos \Lambda - \frac{A}{2} - \frac{A^2}{8\cos \Lambda} + \frac{6(h_{ac,w} - h)\sin \Lambda}{A} \right] \right\}$$
(6.3)

where A is the wing aspect ratio,  $\Lambda$  is the wing sweep angle,  $C_L$  is the total lift coefficient,  $h_{ac,w}$  is the location of the wing aerodynamic center in percent chord and h is the location of c.g. in percent chord.

– Due to roll rate, the right wing will move down and thereby see an increase in angle of attack of py/V applied to a wing section located at a distance y from the root. This increase in angle of attack will tilt the lift vector forward on the right wing, while on the left wing the inclination is to the rear. As a result, a differential yawing moment is created and is given by

$$dN_w = -q(cdy)C_l(\frac{py}{V})(2y)$$
(6.4)

or

$$dN_w = -2\frac{p}{V}q(cC_l y^2 dy) \tag{6.5}$$

Thus integrating from 0 to b/2, we obtain the incremental yawing moment due to roll rate p as

$$\Delta N_w = -2\frac{p}{V}q \int_0^{b/2} cC_l y^2 dy$$
 (6.6)

or, in terms of the yawing moment coefficient, we have

$$\Delta C_{N_w} = -2 \frac{p}{SbV} \int_0^{b/2} cC_l y^2 dy$$
 (6.7)

For a linearly tapered wing with taper ratio  $\lambda = c_t/c_o$  and assuming that  $C_l$  is constant across the wing span, we can show that

$$\Delta C_{N_w} = -\frac{C_L}{12} \left( \frac{1+3\lambda}{1+\lambda} \right) \bar{p} \tag{6.8}$$

or the yawing moment coefficient contributed by the wing due to roll rate is given by

$$C_{N_{w,\bar{p}}} = -\frac{C_L}{12} \left( \frac{1+3\lambda}{1+\lambda} \right) \tag{6.9}$$

in terms of the dimensionless roll rate  $\bar{p} = pb/2V$ .

• Fuselage (and Nacelle) Contribution  $N_{fuselage}$ : Yawing moment due to sideslip is a function of the fuselage (or nacelle) volume, length and width as follows,

$$\frac{\partial C_{N_{fuselage}}}{\partial \beta} = C_{N_{fuselage},\beta} = -1.3 \frac{Volume}{Sb} (\frac{D_f}{W_f}) \text{ (per radians)}$$
 (6.10)

where  $W_f$  and  $D_f$  are respectively themaximum widthanddepth of the fuselage. Clearly, the fuselage produces a negative contribution to the lateral stability, i.e., making the vehicle less stable in the yaw axis.

- Vertical Tail Contribution N<sub>verticaltail</sub>: The vertical tail plays as significant a role in the lateral motion
  as the horizontal tail in the longitudinal motion. The effects on the vertical tail due to sideslip, roll rate
  and yaw rate are described below.
  - Due to a sideslip  $\beta$ , the vertical tail produces a side force  $F_v$  which in turn will result in yawing moment about the center of gravity as follows. Note that the side force  $F_v$  is given by

$$F_{v} = -\eta_{v}qS_{v}a_{v}\beta_{v} = -\eta_{v}qS_{v}a_{v}\frac{\partial\beta_{v}}{\partial\beta}\beta = -\eta_{v}qS_{v}a_{v}(1 - \epsilon_{\beta})\beta$$
(6.11)

The yawing moment coefficient produced by this side force is

$$\Delta C_{N_v} = C_{N_{v,\beta}} \beta = -\frac{F_v l_v}{aSb} \tag{6.12}$$

Using equation (6.11), we deduce

$$C_{N_{v,\beta}} = \eta_v \frac{S_v}{S} a_v \frac{l_v}{h} (1 - \epsilon_\beta) \tag{6.13}$$

where  $\eta_v$  is the ratiobetween the dynamic pressure at the verticaltail and the freestream dynamic pressure,  $l_v$  is the distance from the aerodynamic center of the vertical tail to the center of gravity,  $\epsilon_{\beta}$  is the sidewash factor.

- Due to roll rate p, points along the vertical tail will see an increase in angle of attack of py/V, thus we have an incremental yawing moment of,

$$dN_v = q_v(c_v dy) a_v \frac{yp}{V} l_v \tag{6.14}$$

where  $c_v$  is the chord length at a section of the vertical tail located at a distance y from the root. Thus, by integrating from 0 to  $b_v$  (where  $b_v$  is the height of the vertical tail) and dividing by qSb, we obtain the following contribution to yawing moment at the vertical tail due to roll rate,

$$\Delta C_{N_v} = \eta_v \frac{l_v}{Sb} \frac{p}{V} a_v \int_0^{b_v} c_v y dy \tag{6.15}$$

In terms of the dimensionless roll rate  $\bar{p} = pb/2V$ , we have

$$C_{N_{v,\bar{p}}} = 2\frac{\eta_v}{b} \frac{l_v}{Sb} a_v \int_0^{b_v} c_v y dy$$
 (6.16)

– Due to yaw rate r, the vertical tail will have a change in angle of attack of  $\Delta \alpha_v = -rl_v/V$ . An incremental side force of

$$F_v = \eta_v q S_v a_v \frac{r l_v}{V} \tag{6.17}$$

is thereby produced that results in an incremental yawing moment of

$$\Delta N_v = -F_v l_v = -\eta_v q S_v a_v \frac{r l_v}{V} l_v \tag{6.18}$$

In terms of yawing moment coefficient obtained by dividing equation (6.18) by qSb, we have

$$\Delta C_{N_v} = -\eta_v \frac{S_v l_v}{Sh} a_v \frac{l_v r}{V} \tag{6.19}$$

or the yaw damping coefficient  $C_{N_{v,r}}$  primarily due to the vertical tail is given by

$$C_{N_{v,r}} = -\eta_v \frac{S_v l_v}{Sb} a_v \frac{l_v}{V} \tag{6.20}$$

In terms of the dimensionless yaw rate  $\bar{r} = rb/2V$ , we have

$$C_{N_{v,\bar{r}}} = -\eta_v \frac{S_v l_v}{Sb} a_v \frac{l_v}{V} \frac{2V}{b} \tag{6.21}$$

or

$$C_{N_{v,\bar{r}}} = -2\eta_v \frac{S_v l_v}{Sh} a_v \frac{l_v}{h} = -2\eta_v V_V a_v \frac{l_v}{h}$$
(6.22)

where  $V_V = \frac{S_v l_v}{Sb}$  is the vertical tail volume. In Perkins & Hage, additional contribution to the total yawing moment due to yaw rate is included for the differential wing drag as follows,

$$C_{N_{\bar{r}}} = -\frac{C_{D_w}}{4} + C_{N_{v,\bar{r}}} = -\frac{C_{D_w}}{4} - 2\eta_v V_V a_v \frac{l_v}{h}$$
(6.23)

• Propulsion Contribution  $N_{propulsion}$ : As in the longitudinal case, the propulsion system will contribute to the overall yawing moment due to unbalanced thrusts from left and right engines (e.g an engine failure). Also the normal forces exerted at the propeller disc due to sideslip will produce additional yawing moment. In summary, we can write

$$\Delta C_{N_{propulsion}} = \Delta C_{N_{propulsion}} + \Delta C_{N_{propulsion}} \beta$$
 (6.24)

where

$$\Delta C_{N_{propulsion o}} = \frac{-(T_{right} - T_{left})y_p}{qSb}$$
 (6.25)

where  $y_p$  is the distance between the engine and the fuselage centerlines, and

$$\Delta C_{N_{propulsion}} = -N_{prop} \frac{S_{prop} l_p}{Sb} \frac{\partial C_{N_p}}{\partial \beta} (1 - \epsilon_{\beta})$$
(6.26)

where  $N_{prop}$  is the number of propellers,  $l_p$  is the *x*-distance from the propeller to the c.g. (Figure 4.7), the coefficient  $\frac{\partial C_{Np}}{\partial \beta}$  (by symmetry consideration) can be obtained in a similar fashion using Figure 4.8.

• Rudder Contribution  $N_{rudder}$ : As seen in Section 6.3, rudder surface provides an effective way to control the yawing moment of the airplane. The incremental contribution to the yawing moment is governed by equation (6.79) from which we obtain the yawing moment coefficient due to rudder  $\delta_r$  as given in equation (6.80). A major consideration in sizing the rudder is in the case of engine failure. The rudder must have enough authority to trim out the imbalance in yawing moment  $-Ty_p$  due to a failure, for example, of the left engine. Then, since T = D in trim,

$$-Dy_{p} - qSb \eta_{v} V_{V} a_{v} \tau \delta_{r} = 0 \tag{6.27}$$

or

$$\delta_r = -\frac{Dy_p}{qSb\,\eta_v V_V a_v \tau} = -\frac{C_D y_p}{\eta_v b V_V a_v \tau} \tag{6.28}$$

- Aileron Contribution  $N_{aileron}$ : As the ailerons are deflected asymmetrically (and in equal amount) to produce a roll about the x-axis, drag produced at the downward deflected aileron surface is higher than that generated at the upward deflected aileron. In this way, a yawing moment is created by the aileron surfaces when the airplane is in a roll maneuver. For example, in a positive roll maneuver, the left aileron is down and the right aileron is up; thus a negative yawing moment is produced that is adverse to the turn coordination. In some cases, to remove the adverse yaw effect, one can increase the drag of the upward deflected surface by introducing a greater deflection angle at this surface. Other design concepts may be used to generate higher drags (e.g a Frise aileron sticks out into the flow when deflected upward).
- Spoiler Contribution  $N_{spoiler}$ : Aspoileris anaerodynamic device, placedon the upperwing surface, to generate drag; thus it is a speed control effector. The spoilers are sometimes also used for roll control. When deflected (upward), it causes the flow to separate on the upper surface and thereby resulting in a loss of lift. In roll control, deflecting the right spoiler upward will result in a positive rolling moment. At the same time, the drag on the right spoiler would generate a desired positive yawing moment (hence there is no adverse yaw effect when the spoiler is used to roll the airplane). The difficulty in using spoilers as control effectors for roll stabilization is due to the fact its aerodynamic behaviour is highly nonlinear, and besides they are less effective than the aileron control surfaces since they are located near the wing root. In general, a loss in lift will be accompanied by a loss in altitude, and thus may be undesireable.

#### **6.1.2** Contributions to the Rolling Moment

The rolling motion is generally affected by the motion variables in yaw r, sidelip  $\beta$  and roll p. The components that contribute mostly to the rolling moment are the wing, the vertical tail, the ailerons (located on the wing) and the rudder.

The airplane rolling moment about the center of gravity can be written as

$$L = L_{wing} + L_{fuselage} + L_{verticaltail} + L_{aileron} + L_{rudder}$$
(6.29)

In terms of the moment coefficients,

$$C_L = \frac{L}{qSb} = C_{L_{wing}} + C_{L_{fuselage}} + C_{L_{verticaltail}} + C_{L_{aileron}} + C_{L_{rudder}}$$
(6.30)

- Wing Contribution  $L_{wing}$ : The rolling moment produced at the wing is developed primarily from perturbed motions in sideslip  $\beta$ , roll rate p and yaw rate r.
  - Due to sideslip, the rolling moment is primarily obtained from the wing dihedral (depicted by the dihedral angle  $\Gamma > 0$  above the horizontal plane). The rate of change of rolling moment with sideslip,  $C_{L_{\beta}} = \partial C_L/\partial \beta$ , is important to the handling qualities of an airplane. A small *negative* value of  $C_{L_{\beta}}$  is desireable. Too much dihedral will make the airplane hard to fly. When the airplane is in a positive sideslip, the right wing will see an increase in angle of attack of

$$\Delta \alpha = \beta \Gamma V / V = \beta \Gamma \tag{6.31}$$

The opposite change in  $\alpha$  occurs over the left wing. This results in a differential increment in rolling moment,

$$dL_{w} = -2q(cdy)a_{w}\beta\Gamma y = -2qa_{w}\beta\Gamma cydy \tag{6.32}$$

or

$$C_{L_w} = -2a_w \beta \Gamma \frac{\int_0^{b/2} cy dy}{Sb} \tag{6.33}$$

For a linearly tapered wing, we have

$$C_{L_w} = -\frac{a_w}{6} \left(\frac{1+2\lambda}{1+\lambda}\right) \Gamma \beta \tag{6.34}$$

or

$$C_{L_{w,\beta}} = -\frac{a_w}{6} \left(\frac{1+2\lambda}{1+\lambda}\right) \Gamma \tag{6.35}$$

Another effect due to sideslip is derived from a swept-back configured wing. Figure 6.2 shows a swept wing in a positive sideslip. The velocity normal to the right leading edge is  $V \cos(\Lambda - \beta)$  and ontheleftwing  $V \cos(\Lambda + \beta)$ . Let  $C_{l_n}$  bethe section liftcorresponding to the normal velocity  $V \cos(\Lambda - \beta)$  (or  $V \cos(\Lambda + \beta)$ ) and normal chord  $c \cos \Lambda$ ,

Differential lift on the right wing = 
$$dL_R = q \cos^2(\Lambda - \beta)c \cos \Lambda C_{l_n} ds$$
 (6.36)

and

Differential lift on the left wing = 
$$dL_L = q \cos^2(\Lambda + \beta)c \cos \Lambda C_{l_n} ds$$
 (6.37)

The differential rolling moment is simply,

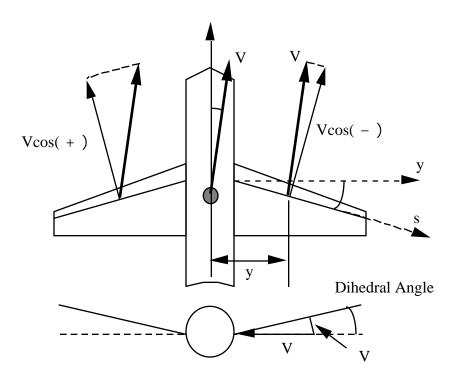


Figure 6.2: Effect of Sweepback on Total Lift and Rolling Moment to Sideslip

$$dL_w = (dL_L - dL_R)y = q[\cos^2(\Lambda + \beta) - \cos^2(\Lambda - \beta)]c \cos \Lambda C_{l_n} y ds$$
 (6.38)

where  $y = s \cos \Lambda$  or  $dy = \cos \Lambda ds$ . Integrating from 0 to b/2 we obtain,

$$L_w = qC_{l_n}[\cos^2(\Lambda + \beta) - \cos^2(\Lambda - \beta)] \int_0^{b/2} cy dy$$
 (6.39)

Note that the incremental lift for a swept wing is

$$dL = \frac{1}{2}\rho(V\cos\Lambda)^2 C_{l_n} c dy \tag{6.40}$$

Hence, integrating over the entire wing span, the total lift for a swept wing is given by

$$L = 2q \cos^2 \Lambda C_{l_n} \int_0^{b/2} c dy = q S C_{l_n} \cos^2 \Lambda$$
 (6.41)

or the wing  $C_L$  and the normal section  $C_{l_n}$  are related by

$$C_L = C_{l_n} \cos^2 \Lambda \tag{6.42}$$

Then the rolling moment coefficient becomes

$$C_{L_w} = \frac{C_L}{\cos^2 \Lambda} [\cos^2(\Lambda + \beta) - \cos^2(\Lambda - \beta)] \frac{\int_0^{b/2} cy dy}{Sb}$$
 (6.43)

If we differentiate with respect to  $\beta$  and evaluate the derivative at  $\beta = 0$ , we obtain

$$C_{L_{w,\beta}} = -4C_L \tan \Lambda \frac{\int_0^{b/2} cy dy}{Sb}$$
 (6.44)

Again for a linear tapered wing, we have

$$C_{L_{w,\beta}} = -\frac{1+2\lambda}{3(1+\lambda)}C_L \tan \Lambda \tag{6.45}$$

Generally, we have

$$C_{L_{w,\beta}} = -f(A,\lambda)C_L \tan \Lambda \tag{6.46}$$

where  $f(A, \lambda)$  is an empirically derived function of aspect ratio A and taper ratio  $\lambda$ . It should be noted that wing placement on the fuselage combined with the cross-flow over the fuselage in sideslip (Figure 6.3) introduces additional factors in the rolling moment due to sideslip, i.e. the rolling coefficient  $C_{L_{m,R}}$ ,

\* High wing:  $\Delta C_{L_{w,\beta}} = -0.00016/deg$ ,

\* Mid-wing:  $\Delta C_{L_{w,\beta}} = 0$ ,

\* Low wing:  $\Delta C_{L_{w,\beta}} = +0.00016/deg$ ,

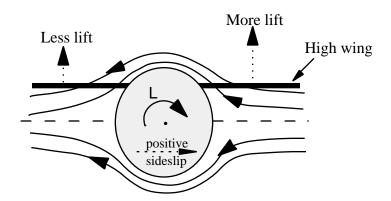


Figure 6.3: Effect of Wing Placement on the Rolling Moment to Sideslip

Due to roll rate, the resulting effect is related to damping in roll. As the airplane rolls, a section
on the right wing located at a distance y from the centerline will experience an increase in angle
of attack of,

$$\Delta \alpha = \frac{py}{V} \tag{6.47}$$

Neglecting induced effects (i.e. tilting of the lift vector), the associated incremental rolling moment is

$$dL_w = -2q(cdy)a_w \frac{py}{V}y (6.48)$$

By integrating from 0 to b/2 and using the nondimensional variable x = y/(b/2), we have

$$C_{L_w} = \frac{L_w}{qSb} = -\frac{a_w A}{2} (\frac{pb}{2V}) \int_0^1 (\frac{c}{b}) x^2 dx$$
 (6.49)

where  $A = b^2/S$  is the wing aspect ratio. For a linearly tapered wing, we have

$$C_{L_w} = -\frac{a_w}{12} \frac{1+3\lambda}{1+\lambda} \bar{p} \tag{6.50}$$

or

$$C_{L_{w,\bar{p}}} = -\frac{a_w}{12} \frac{1+3\lambda}{1+\lambda} \tag{6.51}$$

where the dimensionless roll rate is  $\bar{p} = pb/2V$ .

The aileron control can be used to produce constant roll rate in steady-state. It is obtained from the following equation,

$$C_{L_{\delta_a}}\delta_a + C_{L_{\bar{p}}}\bar{p} = 0 \tag{6.52}$$

or

$$\bar{p} = -\frac{C_{L_{\delta_a}}}{C_{L_{\bar{p}}}} \delta_a \tag{6.53}$$

- Due to yaw rate, the left wing will see a higher velocity than the right wing which is retracting away from the forward motion. Assuming that the wing is operating at a constant  $C_L$  and the section lift  $C_l$  is constant and equals to  $C_L$ . Then a differential rolling moment is produced from the imbalance in dynamic pressure from the two sides; namely

$$dL_{w} = \frac{1}{2}\rho[(V + ry)^{2} - (V - ry)^{2}]C_{L}cdyy$$
(6.54)

$$dL_w = 2\rho VrC_L c y^2 dy (6.55)$$

Integrating from 0 to b/2, and assuming a straight wing with no taper, we obtain

$$L_w = 2\rho VrC_L c \int_0^{b/2} y^2 dy$$
 (6.56)

or

$$L_w = \frac{\rho c C_L r V b^3}{12} = \frac{1}{3} q S b \, \bar{r} C_L \tag{6.57}$$

Therefore,

$$C_{L_w} = \frac{C_L}{3}\bar{r} \tag{6.58}$$

or

$$C_{L_{w,\bar{r}}} = \frac{dC_{L_w}}{d\bar{r}} = \frac{C_L}{3} \tag{6.59}$$

For a linearly tapered wing, we derive

$$C_{L_{w,\bar{r}}} = \frac{C_L}{6} \frac{1+3\lambda}{1+\lambda} \tag{6.60}$$

- Fuselage Contribution  $L_{fuselage}$ : In general, there is no contribution of the fuselage to the rolling moment, i.e.  $L_{fuselage} = 0$ .
- Vertical Tail Contribution  $L_{verticaltail}$ :

– Due to yaw rate, the angle of sideslip is decreased by  $rl_v/V$ . Thus an increment in side force at the tail is

$$F_v = \eta_v q S_v a_v \frac{r l_v}{V} \tag{6.61}$$

And the resulting rolling moment is

$$\Delta L_v = F_v z_v = \eta_v q S_v a_v \frac{r l_v}{V} z_v = 2\eta_v q S_v a_v \frac{r b}{2V} \frac{l_v}{h} z_v = 2\eta_v q S_v a_v \frac{l_v}{h} z_v \bar{r}$$
 (6.62)

where  $z_v$  is the distance of the aerodynamic center of the vertical tail to the axis of rotation (x-axis) and  $\bar{r} = \frac{rb}{2V}$  is the dimensionless yaw rate variable.

Dividing equation (6.62) by qSb, we have

$$C_{L_{v,\bar{r}}} = 2\eta_v \frac{S_v l_v}{Sh} a_v \frac{z_v}{h} = 2\eta_v V_V a_v \frac{z_v}{h}$$
(6.63)

– Due to sideslip, the vertical tail produces a side force  $F_v$  as given in equation (6.11). The rolling moment coefficient produced by this side force is

$$\Delta C_{L_v} = C_{L_{v,\beta}} \beta = \frac{F_v z_v}{qSb} = -\eta_v \frac{qS_v a_v (1 - \epsilon_\beta) \beta z_v}{qSb}$$
(6.64)

From which we deduce

$$C_{L_{v,\beta}} = -\eta_v \frac{S_v}{S} a_v \frac{z_v}{b} (1 - \epsilon_\beta)$$
(6.65)

where the variables  $\eta_v$  and  $\epsilon_{\beta}$  are as exactly those defined for equation (6.11).

• Aileron Contribution  $L_{aileron}$ : Aileroncontrols are effective in the generation of rolling moment due to its location from the axis of rotation (i.e. x-axis). As the right aileron is deflected, there is an increase in sectional lift per unit span produced on the right side. It is given by

$$dL = qca_w dy \tau \delta_{a_R} \tag{6.66}$$

where  $\tau$  is the ailer one ffectiveness (See Figure 9-15 of Perkins & Hage). This results in an incremental change in the rolling moment as follows,

$$dL = -qca_{w}dy\tau\delta_{a_{R}}y\tag{6.67}$$

Combining with the contribution from the left aileron, we have

$$dL = -qca_{w}dy \tau (\delta_{a_{R}} - \delta_{a_{L}})y \tag{6.68}$$

Integrating over the spanwise length of the aileron, we obtain

$$L = -qa_w \tau (\delta_{a_R} - \delta_{a_L}) \int_{y_1}^{y_2} cy dy$$
(6.69)

In dimensionless form, where we define x = y/(b/2), the above equation (6.69) becomes

$$C_L = \frac{L_{\delta_a}}{qSb} = -\frac{1}{4} a_w \tau \delta_a A \int_{x_1}^{x_2} \frac{c}{b} x dx$$
 (6.70)

where  $A = b^2/S$  is the wing aspect ratio and  $\delta_a = \delta_{a_R} - \delta_{a_L}$ . Again for a simple linearly tapered wing, we obtain

$$C_{L_{\delta_a}} = -a_w \tau \frac{3(x_2^2 - x_1^2) - 2(1 - \lambda)(x_2^3 - x_1^3)}{12(1 + \lambda)}$$
(6.71)

• Rudder Contribution  $L_{rudder}$ : As in equation (6.79) for the yawing moment due to rudder, the rudder when deflected will also produce a rolling moment,

$$\Delta L_{\delta_r} = z_v \eta_v q S_v a_v \tau \delta_r \tag{6.72}$$

or, the rolling moment coefficient with respect to  $\delta_r$  is given by

$$C_{L_{\delta_r}} = \eta_v \frac{S_v z_v}{Sb} \tau a_v \tag{6.73}$$

### **6.2** Directional Stability (Weathercock Stability)

Figure 6.4 shows an airplane at a positive sideslip angle  $\beta > 0$  or v > 0. The positive yawing moment N is defined according to the right-hand rule as shown. The non-dimensional yawing moment coefficient is given by

$$C_N = \frac{N}{\frac{1}{2}\rho V^2 Sb} \tag{6.74}$$

The change of yawing moment with respect to sideslip  $\beta$  is defined as

$$C_{N_{\beta}} = \frac{\partial C_N}{\partial \beta} \tag{6.75}$$

This quantity has the same significance as the coefficient  $C_{M_{\alpha}}$ . However, for stability we see that a *positive* yawing moment would be required to bring  $\beta$  back to zero. Therefore, we expect  $C_{N_{\beta}}$  to be *positive* for directional static stability.

Note that when the airplane has a positive sides lip the velocity vector is no longer in the plane of symmetry. There exists a yawing moment produced by the fuse lage and by the side force on the vertical tail. The airplane is directional stable if  $C_{N_{\beta}} > 0$ . Usually the yawing moment due to the fuse lage is destabilizing but its effect is small compared to the stabilizing moment contributed by the vertical tail. However, in most situation, the vertical tail is not sized by any consideration of static stability. Instead the minimum tail size is determined by controllability requirements in the event of an asymmetric engine failure or according to flying quality requirements.

If the aerodynamic center of the vertical tail is located a distance of  $l_v$  behind the center of gravity. Then

$$N = \eta_v q S_v l_v a_v (1 - \epsilon_\beta) \beta \tag{6.76}$$

or

$$C_N = \frac{N}{aSb} = \eta_v \frac{S_v}{S} \frac{l_v}{b} a_v (1 - \epsilon_\beta) \beta \tag{6.77}$$

and thus

$$C_{N_{\beta}} = \eta_t V_V a_v (1 - \epsilon_{\beta}) \tag{6.78}$$

where  $V_V = S_v l_v / Sb$  is the vertical tail volume,  $\epsilon_{\beta}$  is the sidewash factor (difficult to estimate),  $a_v$  is the lift-curve slope for the vertical tail.

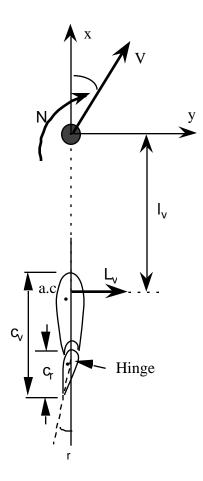


Figure 6.4: Airplane with a Positive Sideslip

#### **6.3** Directional Control

Effective control of yawing moment is provided by the rudder, a movable surface hinged to the vertical stabilizer. Incremental yawing moment created by the rudder is

$$\Delta N = -l_v \Delta L_v \tag{6.79}$$

where  $\Delta L_v = \eta_t q S_v a_v \tau \delta_r$  and the rate of change of  $C_N$  with respect to  $\delta_r$  is given by

$$C_{N_{\delta_r}} = -\eta_v V_V a_v \tau \tag{6.80}$$

where  $\tau$  is the effective factor which depends on the ratio of  $c_r/c_v$  (For example, see Figure 5-33 of Perkins & Hage).

### 6.4 Roll Stability

Rollmomentisgenerated by asymmetric deflection of the right and left ailerons. When an airplane is initially perturbed in roll, and without the use of the aileron controls, there is no physical mechanism to provide a restoring moment. Thus in general  $C_{L_{\phi}}$  is always zero and the concept of static stability does not exist in roll. Or we can also say that the airplane simply possesses neutral static stability in roll. Recall that the rolling moment coefficient is defined as

$$C_L = \frac{L}{qSb} \tag{6.81}$$

Dynamic stability in roll motion is governed by the roll damping term in the stability derivative  $C_{L_{\bar{p}}}$  or the non-dimensional stability derivative  $C_{L_{\bar{p}}}$ . This term is always negative thereby providing positive roll damping.

### 6.5 Roll Control

Roll control is provided primarily by the asymmetric deflection of the left and right aileron surfaces. Effectiveness of the aileron control is determined by the rolling moment coefficient  $C_{L_{\delta_a}}$  derived in equation (6.71). A small contribution in roll control can be derived from the rudder control as defined by the term  $C_{L_{\delta_r}}$  given in equation (6.73). In some airplane, additional roll control may be derived from asymmetric deflection of the spoilers.

## **Chapter 7**

## **Review of Rigid Body Dynamics**

In general a deformable body of finite dimensions may be regarded as being composed of an *infinite* number of particles, thus the system possesses an infinite number of degrees of freedom. This case would apply to a deformable airplane configuration if we take into consideration structural flexibility. However, in this course we consider the airplane to be a *rigid* body with a given mass and moments of inertia. It should be noted that for a rigid body, the system undergoes <u>no</u> deformation and should possess only 6 degrees of freedom, namely 3 translations and 3 rotations.

To describe completely the motion of a rigid body, it is convenient to use:

- 3 translations of a certain point in the rigid body and
- 3 rotations about that point.

A system of axes attached to the body are called *body axes*. As shown in Figure 7.1, the motion of the body can be described by

- 1. Translation of the origin O' of the body axes and
- 2. Rotation of the axes with respect to the inertial space.

It should be noted that velocity of any point P in the rigid body is given by

$$\mathbf{V}_P = \mathbf{V}_{O'} + \omega \times \mathbf{r}_P \tag{7.1}$$

Similarly, acceleration of a point P in the rigid body is given by

$$\mathbf{a}_P = \mathbf{a}_{O'} + \dot{\omega} \times \mathbf{r}_P + \omega \times (\omega \times \mathbf{r}_P) \tag{7.2}$$

To develop the dynamical equations for a rigid body, we need first to define its linear and angular momentum. Consider Figure 7.1 where OXYZ is the inertial reference axes and O'xyz corresponds to a set of axes attached to the rigid body. Note that the origin O' does not necessarily coincide with the body center of mass C. We note the following,

• Mass of the rigid body m is given by

$$m = \int_{Body} dm \tag{7.3}$$

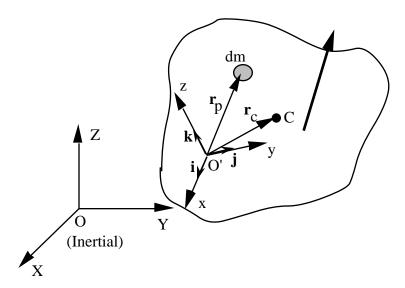


Figure 7.1: Motion of a Rigid Body

• The mass center C is defined as

$$\mathbf{r}_C = \frac{1}{m} \int_{Body} \mathbf{r}_P dm \tag{7.4}$$

Note that if the origin O' coincides with the center of mass C, we have  $\mathbf{r}_C = \mathbf{0}$ .

• The linear momentum of a rigid body is defined as

$$\mathbf{p} = \int_{Body} \mathbf{V}_P dm = \int_{Body} (\mathbf{V}_{O'} + \omega \times \mathbf{r}_P) dm$$
 (7.5)

or

$$\mathbf{p} = \mathbf{V}_{O'} \int_{Body} dm + \omega \times \int_{Body} \mathbf{r}_P dm \tag{7.6}$$

or

$$\mathbf{p} = m(\mathbf{V}_{C'} + \omega \times \mathbf{r}_C) = m\mathbf{V}_C \tag{7.7}$$

where  $V_C$  is the velocity of the center of mass C. Thus the linear momentum of a rigid body is equal to the product of the total mass m and the velocity of the mass center  $V_C$ . It should be noted that the above equation (7.7) applies to any body-axis reference with origin O'. In particular, if O' coincides with the mass center C, we have  $\mathbf{r}_C = \mathbf{0}$  and  $\mathbf{V}_{O'} = \mathbf{V}_C$ .

• Now we derive the angular momentum of a rigid body about the origin O'. By definition,

$$\mathbf{H}_{O'} = \int_{Body} \mathbf{r}_P \times \mathbf{V}_P dm \tag{7.8}$$

or

$$\mathbf{H}_{O'} = \int_{Body} \mathbf{r}_P \times (\mathbf{V}_{O'} + \omega \times \mathbf{r}_P) \, dm \tag{7.9}$$

or

$$\mathbf{H}_{O'} = \int_{Body} \mathbf{r}_P dm \times \mathbf{V}_{O'} + \int_{Body} \mathbf{r}_P \times (\omega \times \mathbf{r}_P) dm$$
 (7.10)

or

$$\mathbf{H}_{O'} = m\mathbf{r}_C \times \mathbf{V}_{O'} + \int_{Body} \mathbf{r}_P \times (\omega \times \mathbf{r}_P) \, dm \tag{7.11}$$

From here on, we conveniently locate the origin O' to be at the center of mass C, then  $\mathbf{r}_C = \mathbf{0}$  and equation (7.11) simplifies to the following

$$\mathbf{H}_C = \int_{Body} \mathbf{r}_P \times (\omega \times \mathbf{r}_P) \, dm \tag{7.12}$$

From vector algebra, we have the following identity:  $\mathbf{r} \times (\omega \times \mathbf{r}) = \omega(r \bullet r) - r(\omega \bullet r)$ . Then equation (7.12) becomes

$$\mathbf{H}_{C} = \int_{Body} \omega(\mathbf{r}_{P} \bullet \mathbf{r}_{P}) dm - \int_{Body} \mathbf{r}_{P}(\omega \bullet \mathbf{r}_{P}) dm$$
 (7.13)

In Cartesian coordinates, where

$$\mathbf{r}_P = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{7.14}$$

and

$$\omega = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k} \tag{7.15}$$

we deduce

$$\mathbf{r}_P \bullet \mathbf{r}_P = x^2 + y^2 + z^2 \tag{7.16}$$

and

$$\omega \bullet \mathbf{r}_P = \omega_x x + \omega_y y + \omega_z z \tag{7.17}$$

Substituting into equation (7.13) we obtain

$$\mathbf{H}_C = \int_{Body} \left\{ (x^2 + y^2 + z^2)(\omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}) - (\omega_x x + \omega_y y + \omega_z z)(x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) \right\} dm \quad (7.18)$$

or

$$\mathbf{H}_{C} = \int_{Body} \left\{ \left[ (y^{2} + z^{2})\omega_{x} - xy\,\omega_{y} - xz\,\omega_{z} \right] \mathbf{i} + \left[ (x^{2} + z^{2})\omega_{y} - yx\,\omega_{x} - yz\,\omega_{z} \right] \mathbf{j} + \left[ (x^{2} + y^{2})\omega_{z} - zx\,\omega_{y} - zy\,\omega_{y} \right] \mathbf{k} \right\} dm$$

$$(7.19)$$

Letting  $\mathbf{H}_C = H_x \mathbf{i} + H_y \mathbf{j} + H_z \mathbf{k}$ , then

$$H_x = \int_{Body} \left[ (y^2 + z^2)\omega_x - xy\,\omega_y - xz\,\omega_z \right] dm = I_{xx}\,\omega_x - I_{xy}\,\omega_y - I_{xz}\,\omega_z \tag{7.20}$$

$$H_{y} = \int_{Body} \left[ (x^2 + z^2)\omega_y - xy\,\omega_x - yz\,\omega_z \right] dm = -I_{yx}\,\omega_x + I_{yy}\,\omega_y - I_{yz}\,\omega_z \tag{7.21}$$

$$H_z = \int_{Body} \left[ (x^2 + y^2)\omega_z - zx\,\omega_x - zy\,\omega_y \right] dm = -I_{zx}\,\omega_x - I_{zy}\,\omega_y + I_{zz}\,\omega_z \tag{7.22}$$

We can define an inertia matrix **I** to be of the following form,

$$\mathbf{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$
(7.23)

Notice that the matrix **I** is a symmetric, positive definite (i.e nonsingular) matrix whose elements have units of  $(ML^2)$ . The angular momentum equation in (7.13) now can be re-written as

$$\mathbf{H}_C = \mathbf{I}\omega \tag{7.24}$$

where the vector  $\omega$  is the angular velocity vector of the rigid body with components  $(\omega_x, \omega_y, \omega_z)$ . The inertia matrix **I** is a <u>constant</u> (i.e., not time-varying) matrix since it is defined in the body-axis O'xyz. Now we can proceed to the development of the equations of motion for a rigid body from Newton's laws.

### 7.1 Force Equations

We have from Newton's laws,

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{7.25}$$

or

$$\frac{d}{dt}(m\mathbf{V}_C) = \mathbf{F} \tag{7.26}$$

where  $\mathbf{p} = m\mathbf{V}_C$  is the linear momentum of the body, m is the total body mass,  $\mathbf{V}_C$  is the velocity of the center of mass. Since the rigid body has an angular velocity  $\omega$ , then equation (7.26) becomes

$$m \left( \frac{d\mathbf{V}_C}{dt} \right) \Big|_{OXYZ} = m \left[ \left( \frac{d\mathbf{V}_C}{dt} \right) \Big|_{O'xyz} + \omega \times \mathbf{V}_C \right] = \mathbf{F}$$
 (7.27)

Let the velocity of the center of mass  $\mathbf{V}_C = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$  and the angular velocity of the rigid body be  $\omega = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$  where  $\omega_x = p$ ,  $\omega_y = q$  and  $\omega_z = r$ . Then the force equations of motion of a rigid body airplane are given by

• Along the body O'x direction:

$$m(\dot{u} - rv + qw) = F_x = X$$
 force component (7.28)

• Along the body O'y direction:

$$m(\dot{v} - pw + ru) = F_v = Y \text{ force component}$$
 (7.29)

• Along the body O'z direction:

$$m(\dot{w} - qu + pv) = F_z = Z$$
 force component (7.30)

The force components X, Y and Z on the right-hand side of the above equations are due to gravitional force, aerodynamic forces and propulsion forces. We will examine these in the next section. Let's now proceed to the equations of motion for the rotational degrees of freedom.

### **7.2** Moment Equations

With Newton's laws applying to the angular momentum, we have

$$\left. \left( \frac{d\mathbf{H}_C}{dt} \right) \right|_{OXYZ} = \left. \left( \frac{d\mathbf{H}_C}{dt} \right) \right|_{O'xyz} + \omega \times \mathbf{H}_C = \mathbf{M}$$
 (7.31)

Using equation (7.24), equation (7.31) becomes

$$\mathbf{I}\dot{\omega} + \omega \times \mathbf{I}\omega = \mathbf{M} \tag{7.32}$$

Using the above definitions for  $\omega$  and the inertia matrix **I**, equation (7.32) can be written in the following form,

• About the body O'x direction:

$$I_{xx}\dot{p} - (I_{yy} - I_{zz})qr - I_{yz}(q^2 - r^2) - I_{zx}(\dot{r} + pq) - I_{xy}(\dot{q} - rp) = M_x = L$$
(7.33)

• About the body O'y direction:

$$I_{yy}\dot{q} - (I_{zz} - I_{xx})rp - I_{zx}(r^2 - p^2) - I_{xy}(\dot{p} + qr) - I_{yz}(\dot{r} - pq) = M_y = M$$
 (7.34)

• About the body O'z direction:

$$I_{zz}\dot{r} - (I_{xx} - I_{yy})pq - I_{xy}(p^2 - q^2) - I_{yz}(\dot{q} + rp) - I_{zx}(\dot{p} - qr) = M_z = N$$
 (7.35)

The moment components L, M and N on the right-hand side of the above equations are due to aerodynamic forces and propulsion forces. We will examine these in the next section. Note that there is no contribution from the gravitational force since these moments are taken about the center of gravity.

### 7.3 Euler's Angles

The angular velocity components  $\omega_x$  (or p),  $\omega_y$  (or q) and  $\omega_z$  (or r) about the body axes x, y and z cannot be integrated to obtain the corresponding angular displacements about these axes. In other words, the orientation of the rigid body in space is not known until we describe the three rotational degrees of freedom in terms of a set of *independent* coordinates. Of course, such a set is not necessarily unique. One useful set of angular displacements called Euler's angles obtained through successive rotations about three (not necessarily orthogonal) axes as follows.

Formostairplanedynamics, westart with a set of inertial axes OXY Z and perform the following rotations in a particular order (Figure 7.2),

- 1. Rotation about the Z-axis (i.e yaw) through an angle  $\psi \Rightarrow (x_1, y_1, z_1)$ ,
- 2. Rotation about the  $y_1$ -axis (i.e pitch) through an angle  $\theta \Rightarrow (x_2, y_2, z_2)$ ,
- 3. Rotation about the  $x_2$ -axis (i.e roll) through an angle  $\phi \Rightarrow (x_3, y_3, z_3)$ .

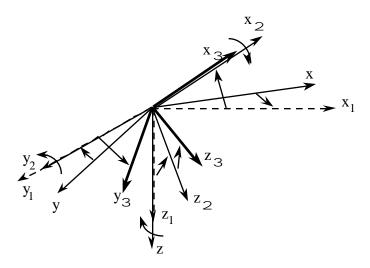


Figure 7.2: Euler's Angle Definition

The Euler's angles for an aircraft are defined as above in terms of  $(\psi, \theta, \phi)$ . At each rotation, components of a vector expressed in the coordinate frame before and after the rotation are related through a rotation matrix. Namely,

•  $\psi$  Rotation:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
 (7.36)

•  $\theta$  Rotation:

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$
 (7.37)

•  $\phi$  Rotation:

$$\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$
 (7.38)

Notice that all the above rotation matrices are orthogonal matrices and hence nonsingular and invertible. Angular velocity  $\omega$  of the rotating frame attached to the rigid body is given by

$$\omega = \dot{\psi}\mathbf{k} + \dot{\theta}\mathbf{j}_1 + \dot{\phi}\mathbf{i}_2 = p\mathbf{i}_3 + q\mathbf{j}_3 + r\mathbf{k}_3 \tag{7.39}$$

We need to express  $\omega$  in terms of its components in the  $(x_3, y_3, z_3)$  coordinate frame. We use the above results for rotation of coordinate frames to obtain.

$$\dot{\phi} - \dot{\psi}\sin\theta = p \tag{7.40}$$

$$\dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi = q \tag{7.41}$$

and

$$\dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi = r \tag{7.42}$$

7.3. EULER'S ANGLES 81

Equations (7.40)-(7.42) are solved (i.e integrated) to determine the orientation of the vehicle  $(\phi, \theta, \psi)$  from the body angular rates (p, q, r) derived from equations (7.33)-(7.35) when we know the externally applied moment components (L, M, N)

We can use the above transformations given in equations (7.36)-(7.38) to express the gravitational force into the body components as follows.

$$\mathbf{F}_{gra\ vity} = mg\,\mathbf{k} = F_{x,gra\ vity}\,\mathbf{i}_3 + F_{y,gra\ vity}\,\mathbf{j}_3 + F_{z,gra\ vity}\,\mathbf{k}_3 \tag{7.43}$$

The "flat-earth" model is represented by having the gravitational force always pointed along the vector  $\mathbf{k}$ . Solving for the components of the gravitational force along the vehicle body axes, we obtain

$$F_{x,gra\,vity} = -mg\sin\theta\tag{7.44}$$

$$F_{y,gra\,vity} = mg\cos\theta\sin\phi\tag{7.45}$$

and

$$F_{z,gravity} = mg\cos\theta\cos\phi \tag{7.46}$$

These components constitute respectively parts of the force components in the right-hand side of equations (7.28)-(7.30). In general, the three successive rotations in equations (7.36)-(7.38) yield the relationship between the coordinates in the two reference frames following the Euler's angle definition,

$$\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
(7.47)

Similarly, one can express the coordinates (x, y, z) in terms of the coordinates  $(x_3, y_3, z_3)$  by reversing the above sequence of rotations. Namely,

•  $-\phi$  Rotation:

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$
 (7.48)

•  $-\theta$  Rotation:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$
 (7.49)

•  $-\psi$  Rotation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \\ z_3 \end{bmatrix}$$
 (7.50)

This yields the following complete transformation,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos \psi \cos \theta & -\sin \psi \cos \phi + \cos \psi \sin \theta \sin \phi & \sin \psi \sin \phi + \cos \psi \sin \theta \cos \phi \\ \sin \psi \cos \theta & \cos \psi \cos \phi + \sin \psi \sin \theta \sin \phi & -\cos \psi \sin \phi + \sin \psi \sin \theta \cos \phi \\ -\sin \theta & \cos \theta \sin \phi & \cos \theta \cos \phi \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$
(7.51)

The above transformation can be used to determine the aircraft position in terms of its linear velocity V with components (u, v, w) in the body-fixed axis. First we make the observation

$$\mathbf{r} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} = u\mathbf{i}_3 + v\mathbf{j}_3 + w\mathbf{k}_3 \tag{7.52}$$

or

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \cos \psi \cos \theta & -\sin \psi \cos \phi + \cos \psi \sin \theta \sin \phi & \sin \psi \sin \phi + \cos \psi \sin \theta \cos \phi \\ \sin \psi \cos \theta & \cos \psi \cos \phi + \sin \psi \sin \theta \sin \phi & -\cos \psi \sin \phi + \sin \psi \sin \theta \cos \phi \\ -\sin \theta & \cos \theta \sin \phi & \cos \theta \cos \phi \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$
(7.53)

Expanding the above expression, we have

$$\dot{x} = u\cos\psi\cos\theta + v(-\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi) + w(\sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi) \tag{7.54}$$

$$\dot{y} = u \sin \psi \cos \theta + v(\cos \psi \cos \phi + \sin \psi \sin \theta \sin \phi) + w(-\cos \psi \sin \phi + \sin \psi \sin \theta \cos \phi)$$
 (7.55)

and

$$\dot{z} = -u\sin\theta + v\cos\theta\sin\phi + w\cos\theta\cos\phi \tag{7.56}$$

Let's summarize here the equations governing the motion of a rigid body aircraft.

- Linear momentum equations:
  - Along the body O'x direction:

$$\dot{u} = rv - qw - g\sin\theta + \frac{1}{m}(X_{aero} + X_{propulsion})$$
 (7.57)

– Along the body O'y direction:

$$\dot{v} = pw - ru + g\sin\phi\cos\theta + \frac{1}{m}(Y_{aero} + Y_{propulsion})$$
 (7.58)

– Along the body O'z direction:

$$\dot{w} = qu - pv + g\cos\theta\cos\phi + \frac{1}{m}(Z_{aero} + Z_{propulsion})$$
 (7.59)

• Angular momentum equations:

$$\dot{\omega} = \mathbf{I}^{-1}(-\omega \times \mathbf{I}\omega + \mathbf{M}) \tag{7.60}$$

• Equations for the vehicle attitude rates:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$
 (7.61)

or

$$\begin{cases} \dot{\theta} = q \cos \phi - r \sin \phi \\ \dot{\psi} = q \sin \phi \sec \theta + r \cos \phi \sec \theta \\ \dot{\phi} = p + q \sin \phi \tan \theta + r \cos \phi \tan \theta \end{cases}$$
(7.62)

7.3. EULER'S ANGLES 83

- Equations for Earth-relative velocities:
  - *x*-distance:

$$\dot{x} = u\cos\psi\cos\theta + v(-\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi) + w(\sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi)$$
(7.63)

- y-distance:

$$\dot{y} = u \sin \psi \cos \theta + v(\cos \psi \cos \phi + \sin \psi \sin \theta \sin \phi) + w(-\cos \psi \sin \phi + \sin \psi \sin \theta \cos \phi)$$
(7.64)

- Vertical altitude h = -z:

$$\dot{h} = u \sin \theta - v \cos \theta \sin \phi - w \cos \theta \cos \phi \tag{7.65}$$

In the above equations, the external forces F and moments M (about the center of gravity) on the right-hand side remain tobe determined. They are derived from basic aerodynamic and propulsion forces and moments. In the above derivation, we made the following assumptions:

- Rigid airframe
- Flat Earth (i.e gravity is always pointing in the vertical k direction)
- Axes fixed to the body with origin at the center of gravity
- Earth-fixed reference is treated as inertial reference

## **Chapter 8**

## **Linearized Equations of Motion**

### 8.1 Linearized Linear Acceleration Equations

Major contributions to the forces and moments in a flight vehicle are coming from the aerodynamics of wings, body and tail surfaces. It would be difficult to express these in terms of the vehicle motion variables u, v and w. However it is much easier to express them in terms of the vehicle velocity V, angle of attack  $\alpha$  and angle of sideslip  $\beta$ . As shown in Figure 8.1, we can express the linear velocities (u, v, w) directly in terms of V,  $\alpha$ ,  $\beta$  through the following relations:

$$\begin{cases} u = V \cos \beta \cos \alpha \\ v = V \sin \beta \\ w = V \cos \beta \sin \alpha \end{cases}$$
(8.1)

where V is the aircraft velocity,  $\alpha$  is the aircraft angle of attack and  $\beta$  the aircraft sideslip.

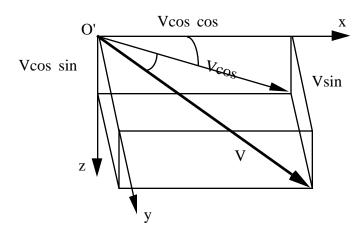


Figure 8.1: Definition of Angle of Attack  $\alpha$  and Sideslip  $\beta$ 

We can rewrite the linear equations of motion given in equations (7.28)-(7.30) as follows,

$$\begin{cases} \dot{u} = rv - qw - g\sin\theta + X/m \\ \dot{v} = pw - ru + g\sin\phi\cos\theta + Y/m \\ \dot{w} = qu - pv + g\cos\theta\cos\phi + Z/m \end{cases}$$
(8.2)

The linear accelerations  $\dot{u}$ ,  $\dot{v}$  and  $\dot{w}$  can be derived in terms of the variables V,  $\beta$  and  $\alpha$  by differentiating equations (8.1) with respect to time t. We obtain

$$\begin{cases} \dot{u} = \dot{V}\cos\alpha\cos\beta - V\dot{\alpha}\sin\alpha\cos\beta - V\dot{\beta}\cos\alpha\sin\beta \\ \dot{v} = \dot{V}\sin\beta + \dot{\beta}V\cos\beta \\ \dot{w} = \dot{V}\sin\alpha\cos\beta + V\dot{\alpha}\cos\alpha\cos\beta - V\dot{\beta}\sin\alpha\sin\beta \end{cases}$$
(8.3)

Or we can re-write equations (8.3) in terms of a linear system of equations,

$$\begin{bmatrix} \cos \alpha \cos \beta & -V \sin \alpha \cos \beta & -V \cos \alpha \sin \beta \\ \sin \beta & 0 & V \cos \beta \\ \sin \alpha \cos \beta & V \cos \alpha \cos \beta & -V \sin \alpha \sin \beta \end{bmatrix} \begin{bmatrix} \dot{V} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix}$$
(8.4)

Solving for  $\dot{V}$ ,  $\dot{\alpha}$  and  $\dot{\beta}$ , we obtain

$$\begin{bmatrix} \dot{V} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \cos \alpha \cos \beta & -V \sin \alpha \cos \beta & -V \cos \alpha \sin \beta \\ \sin \beta & 0 & V \cos \beta \\ \sin \alpha \cos \beta & V \cos \alpha \cos \beta & -V \sin \alpha \sin \beta \end{bmatrix}^{-1} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix}$$
(8.5)

or

$$\begin{bmatrix} \dot{V} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \cos \alpha \cos \beta & \sin \beta & \sin \alpha \cos \beta \\ -\frac{\sin \alpha}{V \cos \beta} & 0 & \frac{\cos \alpha}{V \cos \beta} \\ -\frac{\cos \alpha \cos \beta}{V} & \frac{\cos \beta}{V} & -\frac{\sin \alpha \sin \beta}{V} \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix}$$
(8.6)

Substitute equations (8.2) into equations (8.6) and expand into components,

$$\begin{cases} \dot{V} &= -g(\sin\theta\cos\alpha\cos\beta - \cos\theta\sin\phi\sin\beta - \cos\theta\cos\phi\sin\alpha\cos\beta) \\ &+ \frac{X}{m}\cos\alpha\cos\beta + \frac{Y}{m}\sin\beta + \frac{Z}{m}\sin\alpha\cos\beta \\ \dot{\alpha} &= q - p\cos\alpha\tan\beta - r\sin\alpha\tan\beta + g\frac{\cos\theta\cos\phi\cos\alpha + \sin\theta\sin\alpha}{V\cos\beta} \\ &- \frac{X}{m}\frac{\sin\alpha}{V\cos\beta} + \frac{Z}{m}\frac{\cos\alpha}{V\cos\beta} \\ \dot{\beta} &= p\sin\alpha - r\cos\alpha + \frac{g}{V}(\sin\theta\cos\alpha\sin\beta + \cos\theta\sin\phi\cos\beta \\ &- \cos\theta\cos\phi\sin\alpha\sin\beta) - \frac{X}{mV}\cos\alpha\sin\beta + \frac{Y}{mV}\cos\beta - \frac{Z}{mV}\sin\alpha\sin\beta \end{aligned}$$
 (8.7)

where  $\alpha$ ,  $\beta$ ,  $\theta$  and  $\phi$  are the angles of attack, sideslip, pitch and roll respectively, X, Y and Z are the external forces along the x, y and z-body axes, m is the total airplane mass, g is the gravitational acceleration and V is the total velocity.

From equations (8.7), we can obtain a set of linearized equations in terms of the perturbation variables  $\Delta V$ ,  $\Delta \alpha$ ,  $\Delta \beta$ ,  $\Delta p$ ,  $\Delta q$ ,  $\Delta r$ ,  $\Delta X$ ,  $\Delta Y$ ,  $\Delta Z$ ,  $\Delta \theta$  and  $\Delta \phi$  in a symmetric climb condition with

• Linear velocities:

$$V = V_o + \Delta V, \ \alpha = \alpha_o + \Delta \alpha, \ \beta = \beta_o + \Delta \beta, \tag{8.8}$$

• Angular velocities:

$$p = p_o + \Delta p, \ q = q_o + \Delta q, \ r = r_o + \Delta r, \tag{8.9}$$

• Force components:

$$X = X_o + \Delta X, \quad Y = Y_o + \Delta Y, \quad Z = Z_o + \Delta Z \tag{8.10}$$

• Airplane attitude angles:

$$\theta = \theta_o + \Delta\theta, \ \phi = \phi_o + \Delta\phi, \ \psi = \psi_o + \Delta\psi$$
 (8.11)

where  $V_o$  is the constant aircraft trim velocity,  $\alpha_o$  is the trim angle of attack,  $\theta_o$  is the trim airplane pitch attitude,  $X_o$  is the trim force component in the x-direction and  $Z_o$  is the trim force component in the z-direction. For a symmetric climb condition, we have  $\beta_o = p_o = q_o = r_o = Y_o = \phi_o = \psi_o = 0$ . These trim quantities may not be zero for other flight condition (e.g. steady level turn). Note that the variables  $\Delta V$ ,  $\Delta \alpha$ ,  $\Delta \beta$ ,  $\Delta p$ ,  $\Delta q$ ,  $\Delta r$ ,  $\Delta X$ ,  $\Delta Y$ ,  $\Delta Z$ ,  $\Delta \theta$  and  $\Delta \phi$  are perturbation variables about the trim condition. They are always treated as small quantities.

Substituting equations (8.8)-(8.11)into equations (8.7), we can derive the linearized equations of motion governing the perturbed variables  $\Delta V$ ,  $\Delta \beta$  and  $\Delta \alpha$ . The linearization is done by neglecting the higher-order terms (e.g.  $\Delta \beta \Delta \alpha \approx 0$ ,  $\Delta V \Delta \alpha \approx 0$ , etc...) and invoking at the same time small angle approximations (i.e.  $\cos \Delta \beta \approx 1$ ,  $\sin \Delta \beta \approx \Delta \beta$ , etc...). After some lengthy manipulation, the following equations of motion for the perturbation variables  $\Delta V$ ,  $\Delta \alpha$  and  $\Delta \beta$  are derived,

$$\begin{cases}
\Delta \dot{V} = -g\cos(\theta_o - \alpha_o)\Delta\theta + \frac{\cos\alpha_o}{m}\Delta X + \frac{\sin\alpha_o}{m}\Delta Z \\
\Delta \dot{\alpha} = \Delta q - \frac{g}{V_o}\sin(\theta_o - \alpha_o)\Delta\theta - \frac{\sin\alpha_o}{mV_o}\Delta X + \frac{\cos\alpha_o}{mV_o}\Delta Z \\
\Delta \dot{\beta} = \sin\alpha_o\Delta p - \cos\alpha_o\Delta r + \frac{g}{V_o}\cos\theta_o\Delta\phi + \frac{1}{mV_o}\Delta Y
\end{cases}$$
(8.12)

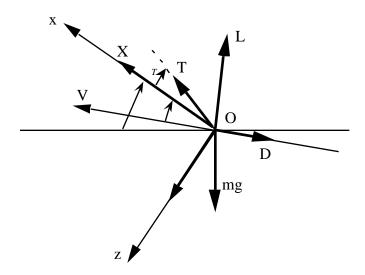


Figure 8.2: X and Z-Force Components in terms of L, D and T

Using Figure (8.2), we can further express the **external** force components X and Z in terms of the lift L, drag D and propulsion force T as follows,

$$\begin{cases} X = L\sin\alpha - D\cos\alpha + T\cos\alpha_T \\ Z = -L\cos\alpha - D\sin\alpha - T\sin\alpha_T \end{cases}$$
 (8.13)

Again we can define  $L = L_o + \Delta L$ ,  $D = D_o + \Delta D$  and  $T = T_o + \Delta T$ . The trim conditions are now determined from force balance with the gravity force  $m\mathbf{g}$ . Namely,

$$\begin{cases} X_o = L_o \sin \alpha_o - D_o \cos \alpha_o + T_o \cos \alpha_T = mg \sin \theta_o \\ Z_o = -L_o \cos \alpha_o - D_o \sin \alpha_o - T_o \sin \alpha_T = -mg \cos \theta_o \end{cases}$$
(8.14)

and

$$\begin{cases} \Delta X = (L_o \cos \alpha_o + D_o \sin \alpha_o) \Delta \alpha + \sin \alpha_o \Delta L - \cos \alpha_o \Delta D + \cos \alpha_T \Delta T \\ \Delta Z = (L_o \sin \alpha_o - D_o \cos \alpha_o) \Delta \alpha - \cos \alpha_o \Delta L - \sin \alpha_o \Delta D - \sin \alpha_T \Delta T \end{cases}$$
(8.15)

Substituting equations (8.15) into equations (8.12), we express the linearized equations in terms of  $\Delta L$ ,  $\Delta D$ ,  $\Delta T$  and  $\Delta Y$ ,

$$\begin{cases}
\Delta \dot{V} = -g\cos(\theta_{o} - \alpha_{o})\Delta\theta - \frac{T_{o}\sin(\theta_{o} + \alpha_{T})}{m}\Delta\alpha - \frac{1}{m}\Delta D + \frac{\cos(\alpha_{o} + \alpha_{T})}{m}\Delta T \\
\Delta \dot{\alpha} = \Delta q - \frac{g}{V_{o}}\sin(\theta_{o} - \alpha_{o})(\Delta\theta - \Delta\alpha) - \frac{T_{o}\cos(\theta_{o} + \alpha_{o})}{mV_{o}}\Delta\alpha \\
- \frac{\sin(\alpha_{o} + \alpha_{T})}{mV_{o}}\Delta T - \frac{1}{mV_{o}}\Delta L
\end{cases}$$

$$\Delta \dot{\beta} = \sin\alpha_{o}\Delta p - \cos\alpha_{o}\Delta r + \frac{g}{V_{o}}\cos\theta_{o}\Delta\phi + \frac{1}{mV_{o}}\Delta Y$$
(8.16)

We also note that

$$\begin{cases}
D_o + mgsin (\theta_o - \alpha_o) - T_o cos(\alpha_o + \alpha_T) = 0 \\
L_o - mgcos(\theta_o - \alpha_o) + T_o sin(\alpha_o + \alpha_T) = 0
\end{cases}$$
(8.17)

Equations (8.16) then reduce to

$$\begin{cases}
\Delta \dot{V} = -g\cos(\theta_{o} - \alpha_{o})\Delta\theta - \frac{L_{o}}{m}\Delta\alpha - \frac{1}{m}\Delta D + \frac{\cos(\alpha_{o} + \alpha_{T})}{m}\Delta T \\
\Delta \dot{\alpha} = \Delta q - \frac{g}{V_{o}}\sin(\theta_{o} - \alpha_{o})\Delta\theta - \frac{D_{o}}{mV_{o}}\Delta\alpha - \frac{1}{mV_{o}}\Delta L - \frac{\sin(\alpha_{o} + \alpha_{T})}{mV_{o}}\Delta T
\end{cases}$$

$$(8.18)$$

$$\Delta \dot{\beta} = \sin\alpha_{o}\Delta p - \cos\alpha_{o}\Delta r + \frac{g}{V_{o}}\cos\theta_{o}\Delta\phi + \frac{1}{mV_{o}}\Delta Y$$

The above equations are the linearized equations of the linear acceleration equations. To complete these equations, one needs to express in details the terms involved in  $\Delta L$ ,  $\Delta D$ ,  $\Delta T$  and  $\Delta Y$  as a function of the vehicle motion variables and their perturbations. In the next section, we proceed to formulate the linearized equations of motion corresponding to the angular velocity components p, q and r.

## 8.2 Linearized Angular Acceleration Equations

According to equations (7.33)-(7.35) and examining the condition related to a steady level climb condition, with the angular velocities defined as

$$\begin{cases}
p = p_o + \Delta p \\
q = q_o + \Delta q \\
r = r_o + \Delta r
\end{cases}$$
(8.19)

and the moments as

$$\begin{cases}
L = L_o + \Delta L \\
M = M_o + \Delta M \\
N = N_o + \Delta N
\end{cases}$$
(8.20)

Notice that for a steady level climb condition,  $p_o = q_o = r_o = 0$  and  $L_o = M_o = N_o = 0$ . Substituting equations (8.19) into equations (7.33)-(7.35) and retaining only the first-order terms in  $\Delta p$ ,  $\Delta q$  and  $\Delta r$ , we obtain

### • About the O'y direction:

$$I_{yy}\Delta\dot{q} - (I_{zz} - I_{xx})\Delta r\Delta p - I_{zx}(\Delta r^2 - \Delta p^2) - I_{xy}(\Delta\dot{p} - \Delta q\Delta r) - I_{yz}(\Delta\dot{r} - \Delta p\Delta q) = \Delta M$$

or, after neglecting all the high-order terms,

$$I_{yy} \Delta \dot{q} - I_{xy} \Delta \dot{p} - I_{yz} \Delta \dot{r} = \Delta M \tag{8.21}$$

Assuming further that the airplane has a symmetry about the O'xz plane then we have  $I_{xy} = I_{yz} = 0$ . Equation (8.21) now simplifies greatly to

$$I_{yy} \Delta \dot{q} = \Delta M$$
 (Pitching equation) (8.22)

#### • About the O'x direction:

$$I_{xx} \Delta \dot{p} - (I_{yy} - I_{zz}) \Delta q \Delta r - I_{yz} (\Delta q^2 - \Delta r^2) - I_{xz} (\Delta \dot{r} + \Delta q \Delta p) - I_{xy} (\Delta \dot{q} - \Delta r \Delta p) = \Delta L$$

or, after neglecting all the high-order terms,

$$I_{xx} \Delta \dot{p} - I_{xz} \Delta \dot{r} = \Delta L$$
 (Rolling equation) (8.23)

Note that in general  $I_{xz} \neq 0$ .

#### • About the O'z direction:

$$I_{zz}\Delta\dot{r} - (I_{xx} - I_{yy})\Delta p\Delta q - I_{xy}(\Delta p^2 - \Delta q^2) - I_{yz}(\Delta\dot{q} + \Delta r\Delta p) - I_{zx}(\Delta\dot{p} - \Delta q\Delta r) = \Delta N$$

or, after neglecting all the high-order terms,

$$I_{zz} \Delta \dot{r} - I_{zx} \Delta \dot{p} = \Delta N$$
 (Yawing equation) (8.24)

In summary, the equations describing the angular accelerations of the vehicle are given by

$$\begin{bmatrix} I_{yy} & 0 & 0 \\ 0 & I_{xx} & -I_{xz} \\ 0 & -I_{zx} & I_{zz} \end{bmatrix} \begin{bmatrix} \Delta \dot{q} \\ \Delta \dot{p} \\ \Delta \dot{r} \end{bmatrix} = \begin{bmatrix} \Delta M \\ \Delta L \\ \Delta N \end{bmatrix}$$
(8.25)

To complete these equations, one needs to express in details the terms involved in  $\Delta M$ ,  $\Delta L$  and  $\Delta N$  as a function of the vehicle motion variables and their perturbations. The vehicle attitudes are described using the Euler's angles. In the next section, equations that describe the vehicle attitudes for small perturbation angles are developed.

### 8.3 Linearized Euler's Angle Equations

From equations (7.40)-(7.42), and assuming the trim conditions as defined in equations (8.8)-(8.11), we have

• For the pitch angle  $\theta$ :

$$\Delta \dot{\theta} \cos \Delta \phi + \Delta \dot{\psi} \cos (\theta_o + \Delta \theta) \sin \Delta \phi = \Delta q$$

Since  $\cos \Delta \phi \approx 1$  and  $\Delta \dot{\psi} \sin \Delta \phi \approx 0$ , then

$$\Delta \dot{\theta} = \Delta q$$
 (Pitch angle equation) (8.26)

• For the yaw (heading) angle  $\psi$ :

$$\Delta \dot{\psi} \cos(\theta_o + \Delta \theta) \cos \Delta \phi - \Delta \dot{\theta} \sin \Delta \phi = \Delta r$$

or

$$\Delta \dot{\psi}(\cos\theta_o - \sin\theta_o \Delta\theta) = \Delta r$$

Further simplification yields

$$\Delta \dot{\psi} = \frac{1}{\cos \theta_o} \Delta r \qquad \text{(Yaw angle equation)} \tag{8.27}$$

• For the bank angle  $\phi$ :

$$\Delta \dot{\phi} - \Delta \dot{\psi} \sin \left(\theta_o + \Delta \theta\right) = \Delta p$$

or

$$\Delta \dot{\phi} - \Delta \dot{\psi} (\sin \theta_o + \cos \theta_o \Delta \theta) = \Delta p$$

Again, ignoring the high-order terms,

$$\Delta \dot{\phi} - \sin \theta_o \Delta \dot{\psi} = \Delta p$$
 (Bank angle equation) (8.28)

Using equation (8.27), the above equation simplifies to

$$\Delta \dot{\phi} = \Delta p + \tan \theta_o \Delta r \tag{8.29}$$

#### 8.4 Forces and Moments in terms of their Coefficient Derivatives

In this section, the lift, drag and propulsion forces are expressed in terms of the motion variables and their perturbations; similarly, for moments about the vehicle axes. Results are represented in terms of the vehicle stability derivatives.

Let's define the following dimensionless motion related variables,

$$\begin{cases}
\bar{p} = pb/(2V_o) \\
\bar{q} = qc/(2V_o) \\
\bar{r} = rb/(2V_o)
\end{cases}$$

$$\bar{V} = V/V_o \\
\bar{\alpha} = \dot{\alpha}c/(2V_o) \\
\bar{\beta} = \dot{\beta}b/(2V_o)$$
(8.30)

Equations describing forces due to aerodynamics and propulsion are given below.

#### **8.4.1 Lift Force** *L*

$$L = \frac{1}{2}\rho V^2 SC_L = L_o + \Delta L \tag{8.31}$$

where  $C_L$  is the total lift coefficient. At trim, we have

$$L_o = L_{trim} = \frac{1}{2} \rho_o V_o^2 SC_{L_{trim}}$$
(8.32)

And the perturbation in lift  $\Delta L$  is given by expanding equation (8.31) in terms of  $\Delta V$  and  $\Delta C_L$ . Namely

$$\Delta L = \rho_o V_o S C_{L_{trim}} \Delta V + \frac{1}{2} \rho_o V_o^2 S \Delta C_L$$
 (8.33)

Now we can express  $\Delta C_L$  in terms of the vehicle lift coefficient derivatives,

$$\Delta C_{L} = C_{L_{M}} \Delta M + C_{L_{h}} \Delta h + C_{L_{\alpha}} \Delta \alpha + C_{L_{\beta}} \Delta \beta$$

$$+ C_{L_{\bar{p}}} \Delta \bar{p} + C_{L_{\bar{q}}} \Delta \bar{q} + C_{L_{\bar{r}}} \Delta \bar{r}$$

$$+ C_{L_{\bar{\alpha}}} \Delta \bar{\alpha} + C_{L_{\bar{\beta}}} \Delta \bar{\beta}$$

$$+ C_{L_{\delta_{e}}} \Delta \delta_{e} + C_{L_{\delta_{sp}}} \Delta \delta_{sp} + C_{L_{\delta_{a}}} \Delta \delta_{a} + C_{L_{\delta_{r}}} \Delta \delta_{r} + \cdots$$

$$(8.34)$$

Thecoefficients  $C_{L_{\bar{a}}}$  (and likewise  $C_{L_{\bar{\beta}}}$  come about from the fact that the influence of downwash (or sidewash)  $\epsilon$  on the tail angle of attack (or sideslip) is not felt until it has been propagated from the wing (where it was generated) to the tail location traveling at a velocity  $V_o$ . This means that the downwash (or sidewash) can be expressed in terms of the wing angle of attack  $\alpha$  (or sideslip  $\beta$ ) through a time-delay function  $e^{-\tau s}$  where  $\tau = \frac{l_t}{V_o}$ . Namely,

$$\epsilon = \epsilon_o + \frac{\partial \epsilon}{\partial \alpha} e^{-l_t s/V_o} \alpha \tag{8.35}$$

and

$$\epsilon = \epsilon_o + \frac{\partial \epsilon}{\partial \beta} e^{-l_t s/V_o} \beta \tag{8.36}$$

Noting that the exponential function  $e^{-l_t s/V_o}$  can be expanded into

$$e^{\frac{-l_t s}{V_o}} \approx 1 - \frac{l_t s}{V_o} + \frac{1}{2} \left(\frac{l_t s}{V_o}\right)^2 + \dots$$
 (8.37)

Thus, the coefficient  $C_{L_{\bar{\alpha}}}$  is obtained from

$$\Delta L_t(s) = \eta_t q S_t a_t \alpha_t(s)$$

or

$$\Delta L_t(s) = \eta_t q S_t a_t \left( \alpha(s) - \epsilon(s) \right) \tag{8.38}$$

Substituting equations (8.35) and (8.37) into equation (8.38), we otain

$$\Delta L_t(s) = \eta_t q S_t a_t \left( \alpha(s) - \epsilon_o - \frac{\partial \epsilon}{\partial \alpha} e^{-l_t s/V_o} \alpha(s) \right)$$

or

$$\Delta L_t(s) = \eta_t q S_t a_t \left\{ \alpha(s) - \epsilon_o - \frac{\partial \epsilon}{\partial \alpha} \left[ 1 - \frac{l_t s}{V_o} + \frac{1}{2} \left( \frac{l_t s}{V_o} \right)^2 + \dots \right] \alpha(s) \right\}$$

Or re-written in time domain, we have

$$\Delta L_t(t) = \eta_t q S_t a_t \left\{ \alpha(t) - \epsilon_o - \frac{\partial \epsilon}{\partial \alpha} \left[ \alpha(t) - \frac{l_t}{V_o} \dot{\alpha}(t) + \frac{1}{2} \left( \frac{l_t}{V_o} \right)^2 \ddot{\alpha}(t) + \dots \right] \right\}$$
(8.39)

From the above equation (8.39), we can deduce that

$$C_{L_{t,\alpha}} = \eta_t \frac{S_t}{S} a_t \left( 1 - \frac{\partial \epsilon}{\partial \alpha} \right) \tag{8.40}$$

Similarly,

$$C_{L_{t,\dot{\alpha}}} = \eta_t \frac{S_t}{S} a_t \left( \frac{\partial \epsilon}{\partial \alpha} \frac{l_t}{V_o} \right)$$
 (8.41)

and for a better approximation one can also include terms involving  $\ddot{\alpha}$ ,

$$C_{L_{t,\ddot{\alpha}}} = -\eta_t \frac{S_t}{S} a_t \frac{\partial \epsilon}{\partial \alpha} \frac{1}{2} \left(\frac{l_t}{V_o}\right)^2 \tag{8.42}$$

or higher-order time derivatives in  $\alpha$ , such as  $\frac{d^3\alpha}{dt^3}$  etc... In terms of the nondimensional variable  $\dot{\alpha}$ , we have

$$C_{L_{\bar{\alpha}}} = C_{L_{t,\dot{\alpha}}} \frac{2V_o}{c} = \eta_t \frac{S_t}{S} a_t \frac{\partial \epsilon}{\partial \alpha} \frac{2l_t}{c}$$
(8.43)

The same procedure could be applied to the calculation of  $C_{L_{\hat{\delta}}}$  and terms involving derivatives with respect to  $\dot{\beta}$  and higher time derivatives in sideslip  $\beta$ . However, in most circumstances, the effects of sideslip on lift are considered insignificant and can be neglected.

Note that we make use of the following relationship to convert between Mach number M and velocity V derivatives,

$$\frac{\partial C_L}{\partial V} = \frac{\partial C_L}{\partial M} \frac{dM}{dV} = \frac{1}{a} \frac{\partial C_L}{\partial M}$$
(8.44)

and a is the speed of sound.

#### 8.4.2 **Drag Force** D

$$D = \frac{1}{2}\rho V^2 SC_D = D_o + \Delta D \tag{8.45}$$

where  $C_D$  is the total drag coefficient. At trim, we have

$$D_o = D_{trim} = \frac{1}{2} \rho_o V_o^2 SC_{D_{trim}}$$

$$\tag{8.46}$$

And the perturbation in drag  $\Delta D$  is given by expanding equation (8.45) in terms of  $\Delta V$  and  $\Delta C_D$ . Namely

$$\Delta D = \rho_o V_o S C_{D_{trim}} \Delta V + \frac{1}{2} \rho_o V_o^2 S \Delta C_D$$
 (8.47)

Now we can express  $\Delta C_D$  in terms of the vehicle drag coefficient derivatives,

$$\Delta C_{D} = C_{D_{M}} \Delta M + C_{D_{h}} \Delta h + C_{D_{\alpha}} \Delta \alpha + C_{D_{\beta}} \Delta \beta$$

$$+ C_{D_{\bar{p}}} \Delta \bar{p} + C_{D_{\bar{q}}} \Delta \bar{q} + C_{D_{\bar{r}}} \Delta \bar{r}$$

$$+ C_{D_{\bar{\alpha}}} \Delta \bar{\dot{\alpha}} + C_{D_{\bar{\beta}}} \Delta \bar{\dot{\beta}}$$

$$+ C_{D_{\delta_{e}}} \Delta \delta_{e} + C_{D_{\delta_{sp}}} \Delta \delta_{sp} + C_{D_{\delta_{a}}} \Delta \delta_{a} + C_{D_{\delta_{r}}} \Delta \delta_{r} + \cdots$$

$$(8.48)$$

#### **8.4.3 Side-Force** *Y*

$$Y = \frac{1}{2}\rho V^2 SC_Y = Y_o + \Delta Y$$
 (8.49)

where  $C_Y$  is the total side-force coefficient. And the perturbation in side force  $\Delta Y$  is given by expanding equation (8.49) in terms of  $\Delta V$  and  $\Delta C_Y$ . Namely

$$\Delta Y = \rho_o V_o S C_{Y_{trim}} \Delta V + \frac{1}{2} \rho_o V_o^2 S \Delta C_Y = \frac{1}{2} \rho_o V_o^2 S \Delta C_Y$$
 (8.50)

Since intrim,  $Y_{trim} = 0$  or  $C_{Y_{trim}} = 0$ . Now we can express  $\Delta C_Y$  in terms of the vehicle side-force coefficient derivatives,

$$\Delta C_{Y} = C_{Y_{M}} \Delta M + C_{Y_{h}} \Delta h + C_{Y_{\alpha}} \Delta \alpha + C_{Y_{\beta}} \Delta \beta$$

$$+ C_{Y_{\bar{\rho}}} \Delta \bar{p} + C_{Y_{\bar{q}}} \Delta \bar{q} + C_{Y_{\bar{r}}} \Delta \bar{r}$$

$$+ C_{Y_{\bar{\alpha}}} \Delta \bar{\dot{\alpha}} + C_{Y_{\bar{\beta}}} \Delta \bar{\dot{\beta}}$$

$$+ C_{Y_{\delta_{c}}} \Delta \delta_{e} + C_{Y_{\delta_{sn}}} \Delta \delta_{sp} + C_{Y_{\delta_{a}}} \Delta \delta_{a} + C_{Y_{\delta_{r}}} \Delta \delta_{r} + \cdots$$

$$(8.51)$$

#### **8.4.4** Thrust Force T

$$T = \frac{1}{2}\rho V^2 SC_T \tag{8.52}$$

where  $C_T$  is the total thrust coefficient. And the perturbation in thrust  $\Delta T$  is given by expanding equation (8.52) in terms of  $\Delta V$  and  $\Delta C_T$ . Namely

$$\Delta T = \rho_o V_o S C_{T_{trim}} \Delta V + \frac{1}{2} \rho_o V_o^2 S \Delta C_T$$
(8.53)

Now we can express  $\Delta C_T$  in terms of the thrust coefficient derivatives,

$$\Delta C_T = C_{T_{\bar{V}}} \Delta \bar{V} + C_{T_{\alpha}} \Delta \alpha + C_{T_{\delta_{th}}} \Delta \delta_{th} + C_{T_{\bar{\alpha}}} \Delta \bar{\alpha} + \cdots$$
 (8.54)

Similarly, the moments due to the aerodynamic and propulsion forces are given below. However, for simplicity, we assume that the thrust force is assumed to apply at the airplane center of gravity. Hence it has no effects on the vehicle pitching, rolling and yawing moments.

#### **8.4.5** Pitching Moment *M*

$$M = \frac{1}{2}\rho V^2 ScC_M \tag{8.55}$$

where  $C_M$  is the total pitchingmoment coefficient. And the perturbation inpitching moment  $\Delta M$  is given by expanding equation (8.55) in terms of  $\Delta V$  and  $\Delta C_M$ . Note that for a vehicle in trim,  $M_{trim} = C_{M_{trim}} = 0$ . Namely

$$\Delta M = \rho_o V_o ScC_{M_{trim}} \Delta V + \frac{1}{2} \rho_o V_o^2 Sc \Delta C_M = \frac{1}{2} \rho_o V_o^2 Sc \Delta C_M$$
 (8.56)

Now we can express  $\Delta C_M$  in terms of the vehicle pitching moment coefficient derivatives,

$$\Delta C_{M} = C_{M_{M}} \Delta M + C_{M_{h}} \Delta h + C_{M_{\alpha}} \Delta \alpha + C_{M_{\beta}} \Delta \beta$$

$$+ C_{M_{\bar{p}}} \Delta \bar{p} + C_{M_{\bar{q}}} \Delta \bar{q} + C_{M_{\bar{r}}} \Delta \bar{r}$$

$$+ C_{M_{\bar{\alpha}}} \Delta \bar{\alpha} + C_{M_{\bar{\beta}}} \Delta \bar{\beta}$$

$$+ C_{M_{\delta_{c}}} \Delta \delta_{c} + C_{M_{\delta_{cp}}} \Delta \delta_{sp} + C_{M_{\delta_{a}}} \Delta \delta_{a} + C_{M_{\delta_{r}}} \Delta \delta_{r} + \cdots$$

$$(8.57)$$

As in the calculation of  $C_{L_{\tilde{\alpha}}}$ , we can use the same approach to obtain  $C_{M_{\tilde{\alpha}}}$ . Namely,

$$\Delta M_t = -\Delta L_t l_t \tag{8.58}$$

Thus

 $C_{M_{\dot{\alpha}}} = -\frac{l_t}{c} C_{L_{\dot{\alpha}}}$ 

or

$$C_{M_{\dot{\alpha}}} = -\eta_t V_H a_t \left( \frac{\partial \epsilon}{\partial \alpha} \frac{l_t}{V_o} \right) \tag{8.59}$$

In terms of the nondimensional variable  $\dot{\alpha}$ , we have

$$C_{M_{\tilde{\alpha}}} = C_{M_{\hat{\alpha}}} \frac{2V_o}{c} = -\eta_t V_H a_t \frac{\partial \epsilon}{\partial \alpha} \frac{2l_t}{c}$$
(8.60)

#### **8.4.6** Yawing Moment N

$$N = \frac{1}{2}\rho V^2 SbC_N \tag{8.61}$$

where  $C_N$  is the total yawing moment coefficient. And the perturbation in yawing moment  $\Delta N$  is given by expanding equation (8.61) in terms of  $\Delta V$  and  $\Delta C_N$ . Note that for a vehicle in trim,  $N_{trim} = C_{N_{trim}} = 0$ . Namely

$$\Delta N = \rho_o V_o SbC_{N_{trim}} \Delta V + \frac{1}{2} \rho_o V_o^2 Sb \Delta C_N = \frac{1}{2} \rho_o V_o^2 Sb \Delta C_N$$
(8.62)

Now we can express  $\Delta C_N$  in terms of the vehicle yawing moment coefficient derivatives,

$$\Delta C_{N} = C_{N_{M}} \Delta M + C_{N_{h}} \Delta h + C_{N_{\alpha}} \Delta \alpha + C_{N_{\beta}} \Delta \beta$$

$$+ C_{N_{\bar{p}}} \Delta \bar{p} + C_{N_{\bar{q}}} \Delta \bar{q} + C_{N_{\bar{r}}} \Delta \bar{r}$$

$$+ C_{N_{\bar{\alpha}}} \Delta \bar{\alpha} + C_{N_{\bar{\beta}}} \Delta \bar{\beta}$$

$$+ C_{N_{\delta_{e}}} \Delta \delta_{e} + C_{N_{\delta_{sp}}} \Delta \delta_{sp} + C_{N_{\delta_{a}}} \Delta \delta_{a} + C_{N_{\delta_{r}}} \Delta \delta_{r} + \cdots$$

$$(8.63)$$

#### **8.4.7** Rolling Moment *L*

$$L = \frac{1}{2}\rho V^2 SbC_L \tag{8.64}$$

where  $C_L$  is the total rolling moment coefficient. And the perturbation in rolling moment  $\Delta L$  is given by expanding equation (8.64) in terms of  $\Delta V$  and  $\Delta C_L$ . Note that for a vehicle in trim,  $L_{trim} = C_{L_{trim}} = 0$ . Namely

$$\Delta L = \rho_o V_o SbC_{L_{trim}} \Delta V + \frac{1}{2} \rho_o V_o^2 Sb \Delta C_L = \frac{1}{2} \rho_o V_o^2 Sb \Delta C_L$$
 (8.65)

Now we can express  $\Delta C_L$  in terms of the vehicle rolling moment coefficient derivatives,

$$\Delta C_{L} = C_{L_{M}} \Delta M + C_{L_{h}} \Delta h + C_{L_{\alpha}} \Delta \alpha + C_{L_{\beta}} \Delta \beta$$

$$+ C_{L_{\bar{p}}} \Delta \bar{p} + C_{L_{\bar{q}}} \Delta \bar{q} + C_{L_{\bar{r}}} \Delta \bar{r}$$

$$+ C_{L_{\bar{\alpha}}} \Delta \bar{\dot{\alpha}} + C_{L_{\bar{\beta}}} \Delta \bar{\dot{\beta}}$$

$$+ C_{L_{\delta_{e}}} \Delta \delta_{e} + C_{L_{\delta_{sp}}} \Delta \delta_{sp} + C_{L_{\delta_{a}}} \Delta \delta_{a} + C_{L_{\delta_{r}}} \Delta \delta_{r} + \cdots$$

$$(8.66)$$

In the most general cases, dynamics of a flight vehicle are fully described by 9 highly coupled *nonlinear* ordinary differential equations governing the motion variables  $\{V, \alpha, \beta, p, q, r, \theta, \phi, \psi\}$ . Linearization about a trim condition reduces them to a set of 9 highly coupled (but) *linear* ordinary differential equations for the perturbed variables  $\{\Delta V, \Delta \alpha, \Delta \beta, \Delta p, \Delta q, \Delta r, \Delta \theta, \Delta \phi, \Delta \psi\}$ . When further simplification can be achieved by taking into considerationthe airplane *symmetry* and the *decoupled* effects in aerodynamic forces and moments. This usually leads to the separation of the the basic equations of motion of an airplane into two *distinct* sets: one set corresponds to the *longitudinal* motion for the variables  $\{\Delta V, \Delta \alpha, \Delta q, \Delta \theta\}$ , and the other set to the *lateral* motion for the variables  $\{\Delta \beta, \Delta p, \Delta r, \Delta \phi, \Delta \psi\}$ .

## **Chapter 9**

# **Linearized Longitudinal Equations of Motion**

In this chapter, we examine the flight dynamics characteristics associated with motion in the longitudinal axis. The assumptions made in the analysis are that the effects of lateral motion on the aerodynamic and propulsion forces and moments associated with the lift L, drag D and thrust T forces are negligeable. Of course, in the model linearization, we also assume that the motion of the vehicle is undergoing small changes in the variables  $\Delta V$ ,  $\Delta \alpha$ ,  $\Delta q$  and  $\Delta \theta$  along with small inputs in the controls  $\Delta \delta_e$ ,  $\Delta \delta_{sp}$  and  $\Delta \delta_{th}$ .

Simple approximation models for the longitudinal dynamics are developed that provide further insights into the frequency separation between the phugoidand the short-period modes. Time responses of the vehicle motion in the longitudinal axis are also developed illustrating the effectiveness of the controls. Critical design parameters affecting these responses are delineated. Flying qualities of the vehicle are subsequently defined in terms of these fundamental response characteristics.

Following the linearization performed in Chapter 8, the equations of motion governing the longitudinal dynamics are for the motion variables  $\{\Delta V, \Delta \alpha, \Delta q, \Delta \theta\}$ . Thus, in general, it is described by a set of 4 linear ordinary differential equations obtained from equations (8.18), (8.25) and (8.26). Namely,

$$\begin{cases}
\Delta \dot{V} = -g\cos(\theta_o - \alpha_o)\Delta\theta - \frac{L_o}{m}\Delta\alpha - \frac{1}{m}\Delta D + \frac{\cos(\alpha_T + \alpha_o)}{m}\Delta T \\
\Delta \dot{\alpha} = \Delta q - \frac{g}{V_o}\sin(\theta_o - \alpha_o)\Delta\theta - \frac{D_o}{mV_o}\Delta\alpha - \frac{1}{mV_o}\Delta L - \frac{\sin(\alpha_T + \alpha_o)}{mV_o}\Delta T \\
\Delta \dot{q} = \frac{1}{I_{yy}}\Delta M \\
\Delta \dot{\theta} = \Delta q
\end{cases}$$
(9.1)

The variables  $\theta_o$ ,  $\alpha_o$ ,  $V_o$ ,  $L_o$  and  $D_o$  are determined from the trim conditions involving usually the solutions of a set of nonlinear algebraic equations.

Using expressions for  $L_o$ ,  $\Delta L$ ,  $D_o$ ,  $\Delta D$  and  $\Delta T$  as derived in equations (8.32), (8.33), (8.46), (8.47) and

(8.53), equations (9.1) become

$$\begin{cases}
\Delta \dot{V} = -g\cos(\theta_{o} - \alpha_{o})\Delta\theta - \frac{1}{m}\frac{1}{2}\rho_{o}V_{o}^{2}SC_{L(trim)}\Delta\alpha - \frac{1}{m}\left\{\rho_{o}V_{o}SC_{D(trim)}\Delta V + \frac{1}{2}\rho_{o}V_{o}^{2}S\right\} \\
\left[\frac{C_{D_{M}}}{a}\Delta V + C_{D_{h}}\Delta h + C_{D_{\alpha}}\Delta\alpha + C_{D_{q}}\Delta\bar{q} + C_{D_{\alpha}}\Delta\bar{\alpha} + C_{D_{\delta_{e}}}\Delta\delta_{e} + C_{D_{\delta_{q}}}\Delta\delta_{sp}\right] \\
+ \frac{\cos(\alpha_{T} + \alpha_{o})}{m}\frac{1}{2}\rho_{o}V_{o}^{2}S\left[C_{T_{V}}\Delta\bar{V} + C_{T_{\alpha}}\Delta\alpha + C_{T_{\delta_{th}}}\Delta\delta_{th}\right] \\
\Delta \dot{\alpha} = \Delta q - \frac{g}{V_{o}}\sin(\theta_{o} - \alpha_{o})\Delta\theta - \frac{1}{mV_{o}}\frac{1}{2}\rho_{o}V_{o}^{2}SC_{D(trim)}\Delta\alpha - \frac{1}{mV_{o}}\left\{\rho_{o}V_{o}SC_{L(trim)}\Delta V + \frac{1}{2}\rho_{o}V_{o}^{2}S\left[\frac{C_{L_{M}}}{a}\Delta V + C_{L_{h}}\Delta h + C_{L_{\alpha}}\Delta\alpha + C_{L_{q}}\Delta\bar{q} + C_{L_{\alpha}}\Delta\bar{\alpha} + C_{L_{\delta_{e}}}\Delta\delta_{e}\right] \\
+ C_{L_{\delta_{sp}}}\Delta\delta_{sp}\right] - \frac{\sin(\alpha_{T} + \alpha_{o})}{mV_{o}}\frac{1}{2}\rho_{o}V_{o}^{2}S\left[C_{T_{V}}\Delta\bar{V} + C_{T_{\alpha}}\Delta\alpha + C_{T_{\delta_{th}}}\Delta\delta_{th}\right] \\
\Delta \dot{q} = \frac{1}{I_{yy}}\frac{1}{2}\rho_{o}V_{o}^{2}Sc\left[\frac{C_{M_{M}}}{a}\Delta V + C_{M_{h}}\Delta h + C_{M_{\alpha}}\Delta\alpha + C_{M_{q}}\bar{\Delta}\bar{q} + C_{M_{\alpha}}\bar{\alpha}\bar{\alpha} + C_{M_{\alpha}}\bar{\alpha}\bar{\alpha} + C_{M_{\delta_{e}}}\Delta\delta_{e} + C_{M_{\delta_{sp}}}\Delta\delta_{sp}\right] \\
\Delta \dot{\theta} = \Delta q$$
(9.2)

For an airplane trimmed at level flight, we have  $\Delta h = 0$ . Using the definition of the nondimensional variables  $\Delta \bar{V}$ ,  $\Delta \bar{q}$  and  $\Delta \bar{\alpha}$  in equations (8.30), we obtain

$$\begin{bmatrix} 1 & \frac{\rho_o V_o Sc}{4m} C_{D_{\bar{\alpha}}} & 0 & 0 \\ 0 & 1 + \frac{\rho_o Sc}{4m} C_{L_{\bar{\alpha}}} & 0 & 0 \\ 0 & -\frac{\rho_o V_o Sc^2}{4I_{yy}} C_{M_{\bar{\alpha}}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta \dot{V} \\ \Delta \dot{\alpha} \\ \Delta \dot{q} \\ \Delta \dot{\theta} \end{bmatrix} = \begin{bmatrix} -\frac{\rho_o V_o S}{2m} \left( 2C_D + C_{D_M} M - C_{T_{\bar{V}}} \cos(\alpha_T + \alpha_o) \right) \\ -\frac{\rho_o S}{2m} \left( 2C_L + C_{L_M} M + C_{T_{\bar{V}}} \sin(\alpha_T + \alpha_o) \right) \\ \frac{\rho_o V_o Sc}{2I_{yy}} C_{M_M} M \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\rho_o V_o^2 S}{2m} \left( C_L - C_{D_\alpha} + C_{T_\alpha} \cos(\alpha_T + \alpha_o) \right) & \frac{\rho_o V_o Sc}{4m} C_{D_{\bar{q}}} & -g \cos(\theta_o - \alpha_o) \\ -\frac{\rho_o V_o S}{2m} \left( C_D + C_{L_\alpha} + C_{T_\alpha} \sin(\alpha_T + \alpha_o) \right) & 1 - \frac{\rho_o Sc}{4m} C_{L_{\bar{q}}} & -\frac{g}{V_o} \sin(\theta_o - \alpha_o) \\ \frac{\rho_o V_o^2 Sc}{2I_{yy}} C_{M_\alpha} & \frac{\rho_o V_o Sc^2}{4I_{yy}} C_{M_{\bar{q}}} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta \alpha \\ \Delta q \\ \Delta \theta \end{bmatrix} +$$

$$\begin{bmatrix} -\frac{\rho_{o}V_{o}^{2}S}{2m}C_{D_{\delta e}} & -\frac{\rho_{o}V_{o}^{2}S}{2m}C_{D_{\delta sp}} & \frac{\rho_{o}V_{o}^{2}S}{2m}C_{T_{\delta th}}\cos(\alpha_{T} + \alpha_{o}) \\ -\frac{\rho_{o}V_{o}S}{2m}C_{L_{\delta e}} & -\frac{\rho_{o}V_{o}S}{2m}C_{L_{\delta sp}} & -\frac{\rho_{o}V_{o}S}{2m}C_{T_{\delta th}}\sin(\alpha_{T} + \alpha_{o}) \\ \frac{\rho_{o}V_{o}^{2}Sc}{2I_{yy}}C_{M_{\delta e}} & \frac{\rho_{o}V_{o}^{2}Sc}{2I_{yy}}C_{M_{\delta sp}} & \frac{\rho_{o}V_{o}^{2}Sc}{2I_{yy}}C_{M_{\delta th}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\delta_{e} \\ \Delta\delta_{sp} \\ \Delta\delta_{th} \end{bmatrix}$$
(9.3)

We can solve for  $\{\Delta \dot{V}, \ \Delta \dot{\alpha}, \ \Delta \dot{q}, \ \Delta \dot{\theta}\}$  as follows,

$$\begin{bmatrix} \Delta \dot{V} \\ \Delta \dot{\alpha} \\ \Delta \dot{q} \\ \Delta \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\rho_{o} V_{o} Sc}{4m} C_{D_{\bar{\alpha}}} & 0 & 0 \\ 0 & 1 + \frac{\rho_{o} Sc}{4m} C_{L_{\bar{\alpha}}} & 0 & 0 \\ 0 & -\frac{\rho_{o} V_{o} Sc^{2}}{4I_{yy}} C_{M_{\bar{\alpha}}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{cases} -\frac{\rho_{o} V_{o} S}{2m} \left( 2C_{L} + C_{L_{M}} M + C_{T_{\bar{V}}} \sin(\alpha_{T} + \alpha_{o}) \right) \\ -\frac{\rho_{o} V_{o} Sc}{2m} \left( 2C_{L} + C_{L_{M}} M + C_{T_{\bar{V}}} \sin(\alpha_{T} + \alpha_{o}) \right) \\ -\frac{\rho_{o} V_{o} Sc}{2I_{yy}} C_{M_{M}} M \end{cases}$$

$$\frac{\rho_{o} V_{o}^{2} S}{2m} \left( C_{L} - C_{D_{\alpha}} + C_{T_{\alpha}} \cos(\alpha_{T} + \alpha_{o}) \right) \qquad \frac{\rho_{o} V_{o} Sc}{4m} C_{D_{\bar{q}}} - g \cos(\theta_{o} - \alpha_{o}) \\ -\frac{\rho_{o} V_{o} Sc}{2m} \left( C_{D} + C_{L_{\alpha}} + C_{T_{\alpha}} \sin(\alpha_{T} + \alpha_{o}) \right) \qquad 1 - \frac{\rho_{o} Sc}{4m} C_{L_{\bar{q}}} - \frac{g}{V_{o}} \sin(\theta_{o} - \alpha_{o}) \\ -\frac{\rho_{o} V_{o}^{2} Sc}{2I_{yy}} C_{M_{\alpha}} \qquad \frac{\rho_{o} V_{o}^{2} Sc}{2I_{yy}} C_{M_{\bar{q}}} \qquad 0 \\ 1 \qquad 0 \qquad 1 \end{cases}$$

$$\begin{bmatrix} \Delta V \\ \Delta \alpha \\ \Delta \theta \\ \Delta \theta \end{bmatrix} +$$

$$\begin{bmatrix} -\frac{\rho_{o} V_{o}^{2} Sc}{2m} C_{D_{\bar{b}e}} - \frac{\rho_{o} V_{o}^{2} Sc}{2m} C_{D_{\bar{b}sp}} - \frac{\rho_{o} V_{o}^{2} Sc}{2m} C_{T_{\bar{b}th}} \cos(\alpha_{T} + \alpha_{o}) \\ -\frac{\rho_{o} V_{o}^{2} Sc}{2m} C_{L_{\bar{b}e}} - \frac{\rho_{o} V_{o}^{2} Sc}{2m} C_{L_{\bar{b}sp}} - \frac{\rho_{o} V_{o}^{2} Sc}{2m} C_{T_{\bar{b}th}} \sin(\alpha_{T} + \alpha_{o}) \\ -\frac{\rho_{o} V_{o}^{2} Sc}{2I_{yy}} C_{M_{\bar{b}e}} - \frac{\rho_{o} V_{o}^{2} Sc}{2I_{yy}} C_{M_{\bar{b}sp}} - \frac{\rho_{o} V_{o}^{2} Sc}{2I_{yy}} C_{M_{\bar{b}th}} \end{bmatrix} \begin{bmatrix} \Delta \delta_{e} \\ \Delta \delta_{sp} \\ \Delta \delta_{th} \end{bmatrix}$$

$$(9.4)$$

In abbreviated notation, we write the above equation in the following form

$$\begin{bmatrix} \Delta \dot{V} \\ \Delta \dot{\alpha} \\ \Delta \dot{q} \\ \Delta \dot{\theta} \end{bmatrix} = F \begin{bmatrix} \Delta V \\ \Delta \alpha \\ \Delta q \\ \Delta \theta \end{bmatrix} + G \begin{bmatrix} \Delta \delta_e \\ \Delta \delta_{sp} \\ \Delta \delta_{th} \end{bmatrix}$$
(9.5)

where the matrices F and G can be deduced from the above equation directly.

Inmostsituations, we have  $C_{M_M} = 0$  and for level flight  $\gamma_o = \theta_o - \alpha_o = 0$ , then the characteristic equation for the longitudinal equation is  $a^{4} + Bs^{3} + Cs^{2} + Ds + E = 0$  where

$$E = -\frac{\rho_o^2 S^2 V_o^2 c}{4mI_{yy}} \left\{ 2C_L + C_{L_M} M + C_{T_{\bar{V}}} \sin(\alpha_T + \alpha_o) \right\} gC_{M_\alpha}$$
 (9.6)

Thus, a necessary condition for the quartic characteristic equation to have stable roots is for the coefficient E to be *positive*. In this case, clearly if the aircraft is statically stable then E will be positive; in another words, if the aircraft is statically unstable (i.e.  $C_{M_{\alpha}} \geq 0$ ) then E is negative and we know with certainty that the aircraft is dynamically unstable. This situation illustrates the fact that a statically unstable airplane is dynamically unstable; however when the airplane isstatically stable, we cannot tell whether it isdynamically stable until we solve for the roots of the quartic characteristic equation. In most problems, we actually do not calculate the characteristic equation explicitly, rather we develop the corresponding system matrix E (defined above) and solve for its eigenvalues (They are then exactly the roots of the characteristic equation).

A MATLAB-mfile toformulate the longitudinal equations of motion is given below for the design model described in Section 11.1.

```
S = 608;
c=15.95;
b=42.8;
Ixx = 28700;
Iyy=165100;
Izz=187900;
Izx=-520;
Ixy=0;Iyz=0;
M = 0.5;
CL=0.20709;
CD=0.01468;
q=32.17095;
Weight=45000;m=Weight/g;
%Lift Coefficient Derivatives
CLq=-17.2322;
CLM=7.45058e-6;
CLalpha=4.8706;
CLalphadot=17.2322;
CLelev=0.572957;
%Drag Coefficient Derivatives
CDq=0;
CDM=0;
CDalpha=0.37257;
CDalphadot=0;
CDelev=4.38308e-2;
%Pitch Moment Coefficient Derivatives
CMq=3.8953;
CMM = -7.05586e - 6;
CMalpha=-0.168819;
CMalphadot=-11.887;
CMelev=-0.695281;
%Thrust Coefficient Derivatives
CTv=0;
CTalpha=0;
alphaT=0;
%Trim angle of attack
alphao=0.18105*pi/180;
thetao=alphao;
%Matrix A
A=[1,rho*Vo*S*c*CDalphadot/4/m,0,0;
   0,1+rho*S*c*CLalphadot/4/m,0,0;
   0,-rho*Vo*S*c^2*CMalphadot/4/Iyy,1,0;
   0,0,0,1];
B1=[-rho*Vo*S/2/m*(2*CD+CDM*M-CTv*cos(alphaT+alphao));
   -rho*S/2/m*(2*CL+CLM*M+CTv*sin(alphaT+alphao));
```

```
rho*Vo*S*c*CMM*M/2/Iyy;0];
B2=[rho*Vo^2*S/2/m*(CL-CDalpha+CTalpha*cos(alphaT+alphao));
    -rho*Vo*S/2/m*(CD+CLalpha+CTalpha*sin(alphaT+alphao));
    rho*Vo^2*S*c*CMalpha/2/Iyy;
    01;
B3=[rho*Vo*S*c*CDq/4/m;
    1-rho*S*c*CLq/4/m;
    rho*Vo*S*c^2*CMq/4/Iyy;
    1];
B4=[-g*cos(thetao-alphao);
    -g/Vo*sin(thetao-alphao);
    0;
    0];
B=[B1,B2,B3,B4];
C=[-rho*Vo^2*S*CDelev/2/m;
   -rho*Vo*S*CLelev/2/m;
   rho*Vo^2*S*c*CMelev/2/Iyy;
%Longitudinal equations of motion
F=inv(A)*B;
G=inv(A)*C;
%Phugoid and Short-Period modes
eigx(F);
%Elevator pulse inputs
xo=G*pi/180;
t1=[0:.5:100];
u=zeros(t1);
y1=lsim(F,G,eye(4),zeros(4,1),u,t1,xo);
t2=[0:.01:10];
u=zeros(t2);
y2=lsim(F,G,eye(4),zeros(4,1),u,t2,xo);
clg;
subplot(221);
plot(t1,y1(:,1));
grid;
xlabel('Time (sec)')
ylabel('Velocity (ft/sec)')
subplot(223);
plot(t1,180*y1(:,4)/pi);
grid;
xlabel('Time (sec)')
ylabel('Pitch Attitude (deg)')
subplot(222);
plot(t2,180*y2(:,2)/pi);
```

```
grid;
xlabel('Time (sec)')
ylabel('Angle of Attack (deg)')
subplot(224);
plot(t2,180*y2(:,3)/pi);
grid;
xlabel('Time (sec)')
ylabel('Pitch Rate (deg/sec)')
%Short-period approximation: Delete the velocity and theta equations
Fsp=F(2:3,2:3);
Gsp=G(2:3,1);
damp(Fsp);
%Elevator pulse inputs
xo=Gsp*pi/180;
t1=[0:.01:10];
u=zeros(t1);
y1=lsim(Fsp,Gsp,eye(2),zeros(2,1),u,t1,xo);
clg;
subplot(211);
plot(t1,y1(:,1));
grid;
xlabel('Time (sec)')
ylabel('Angle of Attack (deg)')
subplot(212);
plot(t1,180*y1(:,2)/pi);
grid;
xlabel('Time (sec)')
ylabel('Pitch Rate (deg/sec)')
pause
%Phugoid approximation
freqphg=g/Vo*sqrt(2)
```

Running this MATLAB m-file generates the following longitudinal models:

$$\begin{bmatrix} \Delta \dot{V} \\ \Delta \dot{\alpha} \\ \Delta \dot{q} \\ \Delta \dot{\theta} \end{bmatrix} = F \begin{bmatrix} \Delta V \\ \Delta \alpha \\ \Delta q \\ \Delta \theta \end{bmatrix} + G \Delta \delta_e$$
 (9.7)

where the matrices F and G are given by

```
F =

-8.1994e-03 -2.5708e+01 0 -3.2171e+01

-1.9451e-04 -1.2763e+00 1.0000e+00 0

6.9573e-04 1.0218e+00 -2.4052e+00 0

0 0 1.0000e+00 0
```

```
G =
-6.8094e+00
-1.4968e-01
-1.4061e+01
```

Characteristic roots of the longitudinal model are given by the eigenvalues of the system matrix F. This can be calculated using the MATLAB function  $\mathbf{damp}(\mathbf{F})$ .

| Eigenva    | alues     | Damping | Frequency(rad/sec)          |
|------------|-----------|---------|-----------------------------|
| -0.0012693 | 0.10392i  | 0.012   | 0.10392 (Phugoid mode)      |
| -0.0012693 | -0.10392i | 0.012   | 0.10392 (Phugoid mode)      |
| -0.68348   | Oi        | 1.000   | 0.68348 (Short-Period mode) |
| -3.0037    | Oi        | 1.000   | 3.0037 (Short-Period mode)  |

where the damping ratio  $\zeta$  of a complex root ( $s = \sigma + j\omega$ ) is defined as follows

$$\zeta = \frac{-\sigma}{\sqrt{\sigma^2 + \omega^2}}$$

Notice that  $-1 \le \zeta \le 1$ . A negative damping ratio means that the motion is dynamically unstable.

In most longitudinal aircraft motion, we can distinguish two basic modes: the *phugoid* mode and the *short-period* mode. The phugoid mode is the one that has the longest time constant and is usually lightly damped. Responses of the aircraft that are significantly affected by the phugoid mode are the velocity and the pitch attitude as seen in Fig. 9.1. There is very little response seen in the angle of attack  $\Delta \alpha$  when the aircraft is excited in the phugoid mode. Note that the period of the velocity  $\Delta V$  and pitch attitude  $\Delta q$  responses is roughly equal to the period of the phugoid mode determined from the approximation  $T_{phugoid} = \sqrt{2\pi} V_o/g = \sqrt{2\pi} 556.2959/32.17095 = 76.8(\text{sec})$  as discussed in Section 9.1. Damping of the phugoid mode is predominantly governed by the drag coefficient  $C_D$  and its derivative  $C_{D_M}$  in the equation for  $\Delta V$ . More precisely, the term  $(2C_D + C_{D_M} M)$  or  $(2C_D + C_{D_V} V_o)$  is the damping factor in the speed equation.

The short-period mode is displayed in the motion of the airplane angle of attack  $\Delta \alpha$  and pitch rate  $\Delta q$  (Fig. 9.1). It has a relatively short time constant (hence the name short-period). In most situation, this mode is relatively well damped (if not, then one must provide stabilization of thismodes using feedback control for flight safety since the pilot cannot control this mode). The short-period mode is usually identified by a pair of complex roots, but in some flight conditions it can be in terms of two real roots as seen in the above example problem where  $s_3 = -0.68$  rad/sec and  $s_4 = -3.0$  rad/sec. In some control design problems, one would like to obtain a simplified second-order dynamic model for the longitudinal aircraft motion that captures the fast motion (i.e. the short-period mode) only.

### 9.1 Phugoid-Mode Approximation

The phugoid frequency  $\omega_{phugoid}$  in a level-flight trim condition (i.e.  $\gamma_o = \theta_o - \alpha_o = 0$ ) can be estimated from the speed and pitch attitude equations where we neglect the variation in  $\Delta\alpha$  (i.e.  $\Delta\alpha = 0$ ) and the drag effects. Furthermore, we assume  $C_{T_{\bar{\nu}}} = 0$  and  $C_{T_{\alpha}} = 0$  and for a fixed elevator  $\Delta\delta_e = 0$ . In this case, the

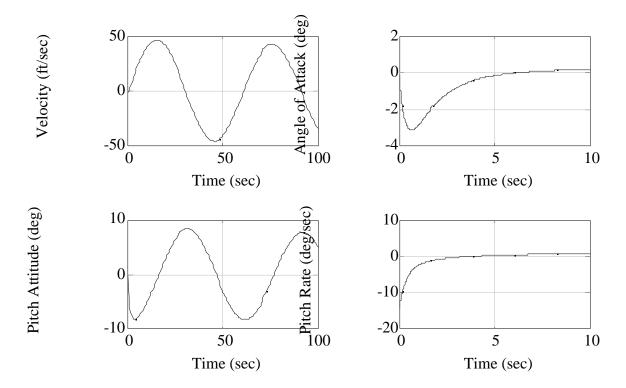


Figure 9.1: Longitudinal Aircraft Responses to a 1-deg Elevator Impulsive Input

motion of the airplane is governed entirely by the exchange of kinetic and potential energies. And we have

$$\begin{cases} \Delta \dot{V} = -g \Delta \theta \\ \Delta \dot{\alpha} = 0 = \Delta q - \frac{\rho_o SC_L}{m} \Delta V \\ \Delta \dot{\theta} = \Delta q = \frac{\rho_o SC_L}{m} \Delta V \end{cases}$$
(9.8)

or

$$\begin{bmatrix} \Delta \dot{V} \\ \Delta \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & -g \\ \frac{\rho_o SC_L}{m} & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta \theta \end{bmatrix}$$
 (9.9)

Characteristic equation of the above equation is simply equal to

$$\det \begin{bmatrix} s & g \\ -\frac{\rho_o SC_L}{m} & s \end{bmatrix} = s^2 + \frac{\rho_o gSC_L}{m} = s^2 + 2\frac{g^2}{V_o^2} = 0$$
 (9.10)

with characteristicroots at  $s_{1,2} = \pm j\sqrt{2}g/V_o$  which correspondtosinusoidal motions in the aircraft velocity and pitch attitude. Thus, the phugoid mode frequency should be roughly equal to  $\omega_{phugoid} = \sqrt{2}g/V_o$  (rad/sec).

Applyingtothegivennumerical example, we have  $\omega_{phugoid} = 0.0818$  rad/secwith aperiod of  $T_{phugoid} = 2\pi/\omega_{phugoid} = 76.8$  sec.

Another interpretation of the phugoid oscillation is as follows. The motion exhibited in the phugoid mode typifies the exchange of potential and kinetic energies of the aircraft. Recall that when the airplane is treated as a point mass m, the total energy is given by

$$\frac{1}{2}mV^2 + mgh = constant (9.11)$$

Differentiating this equation with respect to time, we obtain

$$mV\,\dot{V} + mg\,\dot{h} = 0\tag{9.12}$$

or

$$\dot{V} = -g\frac{\dot{h}}{V} = -g\gamma \tag{9.13}$$

Recall from (way back!) equation (1.9), we have

$$mV\dot{\gamma} = L - W \tag{9.14}$$

where L is the total lift given by

$$L = \frac{1}{2}\rho_o V^2 SC_{L(trim)}$$

and

$$W = mg = \frac{1}{2}\rho_o V_o^2 SC_{L(trim)}$$

Then for small velocity perturbations  $\Delta V$  in  $V = V_o + \Delta V$ , we obtain

$$\Delta \ddot{V} = -g \dot{\gamma} = -g \frac{L - W}{mV} = -g \frac{1}{m(V_o + \Delta V)} \left( \frac{1}{2} \rho_o (V_o + \Delta V)^2 SC_L - \frac{1}{2} \rho_o V_o^2 SC_L \right)$$

or

$$\Delta \ddot{V} \cong -g \frac{1}{mV_o} \left( \rho_o V_o \Delta V S C_L \right) = -\frac{g}{mV_o^2} \left( 2\frac{1}{2} \rho_o V_o^2 S C_L \right) \Delta V$$

$$\Delta \ddot{V} \cong -2\frac{g^2}{V_o^2} \Delta V \tag{9.15}$$

Thus, the perturbed velocity  $\Delta V$  has a second-order harmonic motion with frequency  $\omega_o = \sqrt{2}g/V_o$  (rad/sec).

### 9.2 Short-Period Approximation

A *short-period* approximation model is developed based on the following observations:

- There is a significant frequency separation between the phugoid and short-period modes. Usually an order of magnitude difference in frequency between the phugoid and short-period modes, e.g.  $\omega_{phugoid} = 0.1 \text{ rad/sec}$  and  $\omega_{shortperiod} = 3 \text{ rad/sec}$ .
- Thevelocityoftheaircrafthasnosignificantcomponentsintheshort-periodmode. Inanotherword, the velocitycan be assumed nearlyconstant when theairplane responds to an excitation in the short-period mode.

Thus, the longitudinal equations of motion can be simplified by simply removing (i.e. deleting the variables  $\Delta V$  and  $\Delta \theta$ ) in the original equations. That is, we obtain a set of equations involving only  $\Delta \alpha$  and  $\Delta q$ . Namely,

$$\begin{bmatrix} \Delta \dot{\alpha} \\ \Delta \dot{q} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\rho_o Sc}{4m} C_{L_{\bar{\alpha}}} & 0 \\ -\frac{\rho_o V_o Sc^2 C_{M_{\bar{\alpha}}}}{4I_{yy}} & 1 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -\frac{\rho_o V_o S}{2m} \left( C_D + C_{L\alpha} + C_{T\alpha} \sin(\alpha_T + \alpha_o) \right) & 1 - \frac{\rho_o Sc}{4m} C_{L_{\bar{q}}} \\ \frac{\rho_o V_o^2 Sc}{2I_{yy}} C_{M\alpha} & \frac{\rho_o V_o Sc^2}{4I_{yy}} C_{M_{\bar{q}}} \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta q \end{bmatrix} + \frac{\rho_o V_o Sc^2 C_{M_{\bar{\alpha}}}}{4I_{yy}} C_{M_{\bar{q}}} \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta q \end{bmatrix} + \frac{\rho_o V_o Sc^2 C_{M_{\bar{q}}}}{4I_{yy}} C_{M_{\bar{q}}} \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta q \end{bmatrix} + \frac{\rho_o V_o Sc^2 C_{M_{\bar{q}}}}{4I_{yy}} C_{M_{\bar{q}}} \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta q \end{bmatrix} + \frac{\rho_o V_o Sc^2 C_{M_{\bar{q}}}}{4I_{yy}} C_{M_{\bar{q}}} \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta q \end{bmatrix} + \frac{\rho_o V_o Sc^2 C_{M_{\bar{q}}}}{4I_{yy}} C_{M_{\bar{q}}} C_{M_{\bar{q}}}$$

$$\begin{bmatrix}
-\frac{\rho_o V_o S}{2m} C_{L_{\delta e}} & -\frac{\rho_o V_o S}{2m} C_{L_{\delta sp}} & -\frac{\rho_o V_o S}{2m} C_{T_{\delta th}} \cos(\alpha_T + \alpha_o) \\
\frac{\rho_o V_o^2 Sc}{2I_{yy}} C_{M_{\delta e}} & \frac{\rho_o V_o^2 Sc}{2I_{yy}} C_{M_{\delta sp}} & \frac{\rho_o V_o^2 Sc}{2I_{yy}} C_{M_{\delta th}}
\end{bmatrix} \begin{bmatrix}
\Delta \delta_e \\
\Delta \delta_{sp} \\
\Delta \delta_{th}
\end{bmatrix} \right\}$$
(9.16)

For the above problem, we obtain the following short-period approximation model

$$\begin{bmatrix} \Delta \dot{\alpha} \\ \Delta \dot{q} \end{bmatrix} = \begin{bmatrix} -1.2763 & 1 \\ 1.0218 & -2.4052 \end{bmatrix} \begin{bmatrix} \Delta \alpha (rad) \\ \Delta q \end{bmatrix} + \begin{bmatrix} -0.1497 \\ -14.0611 \end{bmatrix} \Delta \delta_e$$
 (9.17)

where  $\Delta \alpha$ ,  $\Delta \delta_e$  are in radians and  $\Delta q$  in radians/sec. Characteristic roots of the short-period model are given below. They are almost the same as those obtained in the full ( $4^{th}$ -order) longitudinal model.

| Eigenvalues | Damping | Frequency(rad/sec) |
|-------------|---------|--------------------|
| -0.68299    | 1.000   | 0.68299            |
| -2.9985     | 1.000   | 2.9985             |

Figure 9.2 shows the responses of the short-period approximation model to a 1-deg elevator impulse input. Note that these responses of  $\Delta\alpha$  and  $\Delta q$  match closely those obtained for the full (i.e.  $4^{th}$ -order) longitudinal model. In general,  $C_{L_{\alpha}} >> C_D$ ,  $C_{T_{\alpha}} \approx 0$  and  $\frac{\rho_o ScC_{L_{\bar{q}}}}{4m} << 1$ , then the above short-period approximation has the following characteristic equation

$$s^{2} + \frac{\rho_{o} V_{o} Sc^{2}}{2I_{yy}} \left( \frac{I_{yy} C_{L_{\alpha}}}{mc^{2}} - \frac{1}{2} C_{M_{q}} \right) s + \frac{\rho_{o} V_{o}^{2} Sc}{2I_{yy}} C_{L_{\alpha}} \left( -\frac{C_{M_{\alpha}}}{C_{L_{\alpha}}} - \frac{\rho Sc}{4m} C_{M_{\bar{q}}} \right) = 0$$
 (9.18)

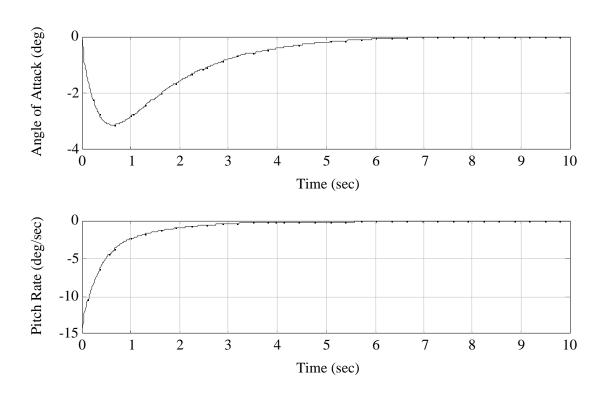


Figure 9.2: Short-Period Approximation Model to a 1-deg Elevator Impulse Input

## **Chapter 10**

## **Linearized Lateral Equations of Motion**

In this chapter, we examine the flight dynamics characteristics associated with motion in the lateral axis. The assumptions made in the analysis are that the effects of longitudinal motion on the aerodynamic and propulsion forces and moments associated with the lift L, drag D and thrust T forces are negligeable. Hence, the motion in the lateral axis is decoupled from the longitudinal dynamics. Of course, in any model linearization, we also assume that the motion of the vehicle is undergoing small changes in the variables  $\Delta\beta$ ,  $\Delta p$ ,  $\Delta r$  and  $\Delta \phi$  along with small inputs in the controls  $\Delta\delta_a$  and  $\Delta\delta_r$ .

Equations governing the lateral motion of an air craft are associated with the motion variables  $\Delta \beta$ ,  $\Delta p$ ,  $\Delta r$ ,  $\Delta \phi$  Hence, the lateral dynamic model is described by a set of 4 linear ordinary differential equations as derived in equations (8.18), (8.25) and (8.28). Namely,

$$\begin{cases}
\Delta \dot{\beta} = \sin \alpha_o \Delta p - \cos \alpha_o \Delta r + \frac{g}{V_o} \cos \theta_o \Delta \phi + \frac{1}{mV_o} \Delta Y \\
\begin{bmatrix}
I_{xx} & -I_{xz} \\
-I_{zx} & I_{zz}
\end{bmatrix}
\begin{bmatrix}
\Delta \dot{p} \\
\Delta \dot{r}
\end{bmatrix} = \begin{bmatrix}
\Delta L \\
\Delta N
\end{bmatrix} \\
\Delta \dot{\phi} = \Delta p + \tan \theta_o \Delta r
\end{cases} (10.1)$$

Using equations (8.50), (8.65) and (8.62) for  $\Delta Y$ ,  $\Delta L$  and  $\Delta N$ , and retaining only the contributions due to lateral motion variables, we obtain

• Sideslip equation for  $\Delta \beta$ :

$$\Delta \dot{\beta} = \sin \alpha_o \Delta p - \cos \alpha_o \Delta r + \frac{g}{V_o} \cos \theta_o \Delta \phi$$

$$+ \frac{1}{mV_o} \frac{1}{2} \rho_o V_o^2 S \left( C_{Y_\beta} \Delta \beta + C_{Y_{\bar{p}}} \Delta \bar{p} + C_{Y_{\bar{p}}} \Delta \bar{r} + C_{Y_{\bar{p}}} \Delta \bar{\beta} + C_{Y_{\delta a}} \Delta \delta_a + C_{Y_{\delta r}} \Delta \delta_r \right) \qquad (10.2)$$
or
$$\left( 1 - \frac{1}{mV_o} C_{Y_{\bar{p}}} \frac{b}{2V_o} \right) \Delta \dot{\beta} = \sin \alpha_o \Delta p - \cos \alpha_o \Delta r + \frac{g}{V_o} \cos \theta_o \Delta \phi$$

$$+ \frac{1}{mV_o} \frac{1}{2} \rho_o V_o^2 S \left( C_{Y_\beta} \Delta \beta + C_{Y_{\bar{p}}} \Delta \bar{p} + C_{Y_{\bar{r}}} \Delta \bar{r} + C_{Y_{\delta a}} \Delta \delta_a + C_{Y_{\delta r}} \Delta \delta_r \right) \qquad (10.3)$$
Dividing by 
$$(1 - \frac{b}{2mV_o^2} C_{Y_{\bar{p}}}), \text{ we obtain}$$

$$\Delta \dot{\beta} = \frac{1}{1 - \frac{b}{2mV_o^2} C_{Y_{\bar{p}}}} \left\{ \frac{\rho_o V_o S}{2m} C_{Y_\beta} \Delta \beta + \left( \sin \alpha_o + \frac{\rho_o b S}{4m} C_{Y_{\bar{p}}} \right) \Delta p \right\}$$

$$+\left(-\cos\alpha_o + \frac{\rho_o bS}{4m}C_{Y_{\bar{r}}}\right)\Delta r + \frac{g}{V_o}\cos\theta_o\Delta\phi + \frac{\rho_o V_o S}{2m}C_{Y_{\delta a}}\Delta\delta_a + \frac{\rho_o V_o S}{2m}C_{Y_{\delta r}}\Delta\delta_r\right\}$$
(10.4)

• Angular velocities  $\Delta p$  and  $\Delta r$ :

$$\begin{bmatrix}
I_{xx} & -I_{xz} \\
-I_{zx} & I_{zz}
\end{bmatrix}
\begin{bmatrix}
\Delta \dot{p} \\
\Delta \dot{r}
\end{bmatrix} =$$

$$\frac{1}{2}\rho_{o}V_{o}^{2}Sb\begin{bmatrix}
C_{L_{\beta}}\Delta\beta + C_{L_{\bar{p}}} \frac{b}{2V_{o}} \Delta p + C_{L_{\bar{r}}} \frac{b}{2V_{o}} \Delta r + C_{L_{\delta a}}\Delta\delta_{a} + C_{L_{\delta r}}\Delta\delta_{r} \\
C_{N_{\beta}}\Delta\beta + C_{N_{\bar{p}}} \frac{b}{2V_{o}} \Delta p + C_{N_{\bar{r}}} \frac{b}{2V_{o}} \Delta r + C_{N_{\delta a}}\Delta\delta_{a} + C_{N_{\delta r}}\Delta\delta_{r}
\end{bmatrix}$$
(10.5)

• Kinematic equation for  $\Delta \phi$ :

$$\Delta \dot{\phi} = \Delta p + \tan \theta_o \Delta r \tag{10.6}$$

Re-arranging the above equations, we obtain a set of 4 linear ordinary differential equations in  $\{\Delta\beta, \Delta p, \Delta r, \Delta\phi\}$  as follows.

$$\begin{bmatrix} \Delta \dot{\beta} \\ \Delta \dot{p} \\ \Delta \dot{r} \\ \Delta \dot{\phi} \end{bmatrix} = \begin{bmatrix} 1 - \frac{b}{2mV_o^2} C_{Y_{\bar{\beta}}} & 0 & 0 & 0 \\ 0 & I_{xx} & -I_{xz} & 0 \\ 0 & 0 & -I_{xz} & I_{zz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{cases} \begin{bmatrix} \frac{\rho_o V_o S}{2m} C_{Y_{\beta}} \\ \frac{\rho_o V_o^2 S b}{2} C_{L_{\beta}} \\ \frac{\rho_o V_o^2 S b}{2} C_{N_{\beta}} \end{bmatrix} \end{cases}$$

$$\sin \alpha_{o} + \frac{\rho_{o}bS}{4m}C_{Y_{\bar{p}}} - \cos \alpha_{o} + \frac{\rho_{o}bS}{4m}C_{Y_{\bar{r}}} \quad \frac{g}{V_{o}}\cos \theta_{o} \\
\frac{\rho_{o}V_{o}Sb^{2}}{4}C_{L_{\bar{p}}} \quad \frac{\rho_{o}V_{o}Sb^{2}}{4}C_{L_{\bar{r}}} \quad 0 \\
\frac{\rho_{o}V_{o}Sb^{2}}{4}C_{N_{\bar{p}}} \quad \frac{\rho_{o}V_{o}Sb^{2}}{4}C_{N_{\bar{r}}} \quad 0 \\
\tan \theta_{o} \quad 0$$
(10.7)

$$\begin{bmatrix} \frac{\rho_o V_o S}{2m} C_{Y_{\delta a}} & \frac{\rho_o V_o S}{2m} C_{Y_{\delta r}} \\ \frac{\rho_o V_o^2 S b}{2} C_{L_{\delta a}} & \frac{\rho_o V_o^2 S b}{2} C_{L_{\delta r}} \\ \frac{\rho_o V_o^2 S b}{2} C_{N_{\delta a}} & \frac{\rho_o V_o^2 S b}{2} C_{N_{\delta r}} \end{bmatrix} \begin{bmatrix} \Delta \delta_a \\ \Delta \delta_r \end{bmatrix}$$

In abbreviated notation, we write the above equation in the following form

$$\begin{bmatrix} \Delta \dot{\beta} \\ \Delta \dot{p} \\ \Delta \dot{r} \\ \Delta \dot{\phi} \end{bmatrix} = F \begin{bmatrix} \Delta \beta \\ \Delta p \\ \Delta r \\ \Delta \phi \end{bmatrix} + G \begin{bmatrix} \Delta \delta_a \\ \Delta \delta_r \end{bmatrix}$$
(10.8)

where the matrices F and G can be deduced from the above equation directly.

Note that we have added an additional equation for the heading variable  $\Delta \psi$  as given in equation (8.27). Namely,

$$\Delta \dot{\psi} = \frac{1}{\cos \theta_o} \Delta r \tag{10.9}$$

A MATLAB-mfile to formulate the lateral equations of motion for the designmodel described in Section 11.1 are given below.

```
clg; clear;
rho=0.00230990;
Vo=556.29559;
S = 608;
c=15.95;
b=42.8;
Ixx = 28700;
Iyy=165100;
Izz=187900;
Izx=-520;
Ixy=0;Iyz=0;
M = 0.5;
CL=0.20709;
CD=0.01468;
g=32.17095;
Weight=45000;m=Weight/g;
%Side Force Coefficient Derivatives
CYp=0;
CYr=0;
CYbeta=-0.97403;
CYbetadot=0;
CYrud=-1.5041e-1;
CYail=-1.1516e-3;
%Yawing Moment Coefficient Derivatives
CNp = -3.3721e - 2;
CNr = -4.0471e - 1;
CNbeta=1.2996e-1;
CNrud=-6.9763e-2;
CNail=2.1917e-3;
%Rolling Moment Coefficient Derivatives
CLp = -0.2;
CLr=0.15099;
CLbeta=-0.13345;
CLrud=-2.3859e-3;
CLail=2.6356e-2;
%Trim angle of attack
alphao=0.18105*pi/180;
thetao=alphao;
%Matrix A
A=[1-b*CYbetadot/(2*m*Vo^2),0,0,0]
   0,Ixx,-Izx,0
   0,-Izx,Izz,0
   0,0,0,1];
B1=[rho*Vo*S*CYbeta/2/m;
    0.5*rho*Vo^2*S*b*CLbeta;
```

```
0.5*rho*Vo^2*S*b*CNbeta;0];
B2=[sin(alphao)+rho*b*S*CYp/4/m;
    0.25*rho*Vo*S*b^2*CLp;
    0.25*rho*Vo*S*b^2*CNp;
    11;
B3=[-cos(alphao)+rho*b*S*CYr/4/m;
    0.25*rho*Vo*S*b^2*CLr;
    0.25*rho*Vo*S*b^2*CNr;
    tan(thetao)];
B4=[q/Vo*cos(thetao);
    0;
    0;
    01;
B=[B1,B2,B3,B4];
C=[rho*Vo*S*CYail/2/m, rho*Vo*S*CYrud/2/m;
   0.5*rho*Vo^2*S*b*CLail, 0.5*rho*Vo^2*S*b*CLrud;
   0.5*rho*Vo^2*S*b*CNail, 0.5*rho*Vo^2*S*b*CNrud;
   0,0];
%Lateral equations of motion
F=inv(A)*B;
G=inv(A)*C;
%Add the heading equation
Fx=[[F,zeros(4,1)];[0,0,1/cos(thetao),0,0]];
Gx = [G; [0,0]];
%Dutch roll, spiral and roll modes
eigx(F);
eigx(Fx);
%Aileron pulse inputs
xo=-Gx(:,1)*pi/180; %Switch sign for + right aileron down
t1=[0:.1:30];
u=zeros(t1);
y1=lsim(Fx,Gx(:,1),eye(5),zeros(5,1),u,t1,xo);
clg;
subplot(221);
plot(t1,180*y1(:,1)/pi);
grid;
xlabel('Time (sec)')
ylabel('Sideslip (deg)')
title('Aileron 1-deg Pulse Input')
subplot(223);
plot(t1,180*y1(:,2)/pi);
grid;
xlabel('Time (sec)')
ylabel('Roll Rate (deg/sec)')
subplot(222);
```

```
plot(t1,180*y1(:,3)/pi);
grid;
xlabel('Time (sec)')
ylabel('Yaw Rate (deg/sec)')
subplot(224);
plot(t1,180*y1(:,4)/pi,t1,180*y1(:,5)/pi);
grid;
xlabel('Time (sec)')
ylabel('Roll/Heading Angles (deg)')
pause
%Rudder pulse inputs
xo=Gx(:,2)*pi/180;
t1=[0:.1:30];
u=zeros(t1);
y1=1sim(Fx,Gx(:,2),eye(5),zeros(5,1),u,t1,xo);
clg;
subplot(221);
plot(t1,180*y1(:,1)/pi);
grid;
xlabel('Time (sec)')
ylabel('Sideslip (deg)')
title('Rudder 1-deg Pulse Input')
subplot(223);
plot(t1,180*y1(:,2)/pi);
grid;
xlabel('Time (sec)')
ylabel('Roll Rate (deg/sec)')
subplot(222);
plot(t1,180*y1(:,3)/pi);
grid;
xlabel('Time (sec)')
ylabel('Yaw Rate (deg/sec)')
subplot(224);
plot(t1,180*y1(:,4)/pi,t1,180*y1(:,5)/pi);
grid;
xlabel('Time (sec)')
ylabel('Roll/Heading Angles (deg)')
```

Running this MATLAB m-file generates the following linearized lateral dynamic model:

$$\begin{bmatrix} \Delta \dot{\beta} \\ \Delta \dot{p} \\ \Delta \dot{r} \\ \Delta \dot{\phi} \\ \Delta \dot{\psi} \end{bmatrix} = F_p \begin{bmatrix} \Delta \beta \\ \Delta p \\ \Delta r \\ \Delta \phi \\ \Delta \psi \end{bmatrix} + G_p \begin{bmatrix} \Delta \delta_a \\ \Delta \delta_r \end{bmatrix}$$
(10.10)

where the matrices  $F_p$  and  $G_p$  are given by

```
Fp =
  -2.7202e-01
                                           5.7830e-02
                                                                  0
                3.1599e-03
                            -1.0000e+00
  -4.3366e+01
              -2.4923e+00
                            1.8964e+00
                                                                  0
                                                    0
   6.5529e+00
              -5.7313e-02
                            -7.7588e-01
                                                                  0
                1.0000e+00
                            3.1599e-03
                                                    0
                                                                  0
                             1.0000e+00
                                                    0
                                                                  0
Gp =
  -3.2161e-04
              -4.2005e-02
   8.5397e+00
              -7.1067e-01
   8.4854e-02
              -3.4512e+00
            0
                         0
                         0
```

Characteristic roots of the lateral model are given by the eigenvalues of the system matrix  $F_p$ . This can be calculated using the MATLAB function  $\mathbf{damp}(\mathbf{F})$ .

| Eigenvalues  |               | Damping | Frequency(rac | d/sec)            |
|--------------|---------------|---------|---------------|-------------------|
| 0.00000e+00  | 0.00000e+00i  | 1.000   | 0.00000e+00   | (Heading mode)    |
| -5.71628e-02 | 0.00000e+00i  | 1.000   | 5.71628e-02   | (Spiral mode)     |
| -3.27498e-01 | 2.73177e+00i  | 0.119   | 2.75133e+00   | (Dutch-roll mode) |
| -3.27498e-01 | -2.73177e+00i | 0.119   | 2.75133e+00   | (Dutch-roll mode) |
| -2.82802e+00 | 0.00000e+00i  | 1.000   | 2.82802e+00   | (Roll mode)       |

There are 4 basic modes associated with the lateral motion:

- The heading mode corresponds to a root at the origin (s=0). This mode is simply associated with the integral of yaw rate for the heading angle  $\Delta \psi$ .
- The spiral mode (s = -0.05716 rad/sec) is a slow mode that is associated with a real root depicting predominantly motion in the roll attitude  $\Delta \phi$ . Its value is significantly affected by the damping in roll from the term  $C_{L_p}$  (i.e. rolling moment due to roll rate). At some flight condition, this mode may even be unstable; since it is a slow mode, the pilot can interact and correct satisfactorily for the spiral instability.
- The Dutch-roll mode ( $s = -0.327498 \pm j2.73177$  rad/sec) is an oscillatory mode with significant components in the yaw  $\Delta r$  and the roll  $\Delta \phi$  variables. This mode did not have adequate damping ( $\zeta = 0.12$ ), and in general flying qualities may dictate the need of a lateral stability augmentation system to improve the Dutch-roll damnping via a yaw-damper feedback control design.
- The roll mode (s = -2.82802 rad/sec) is usually associated with a real root which is located far to the left, i.e. very stable. The motion is predominantly in roll rate  $\Delta p$  and settles down quite quickly.

ShowninFigures(10.1)and(10.2)aretimeresponses in the lateral motion to a separately applied impulse input at the aileron and rudder control surfaces respectively.

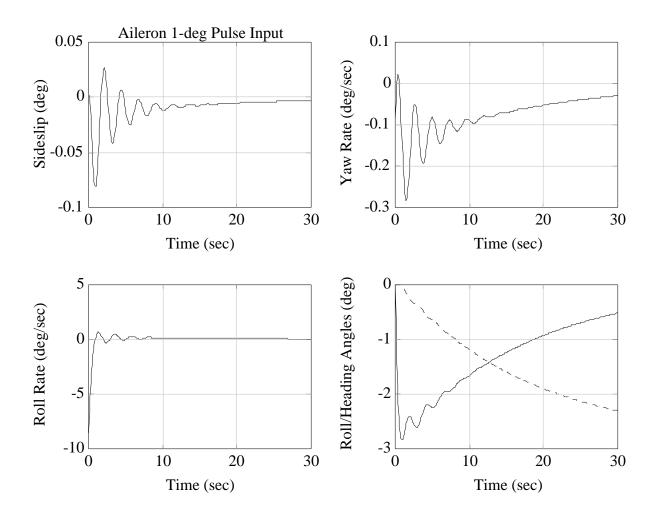


Figure 10.1: Lateral Responses to a 1-deg Aileron Impulse Input

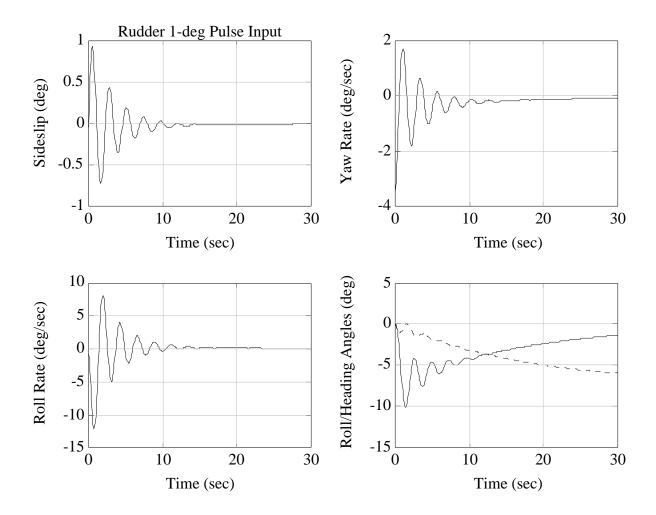


Figure 10.2: Lateral Responses to a 1-deg Rudder Impulse Input

# **Chapter 11**

# Flight Vehicle Models

## 11.1 Generic F-15 Model Data (Subsonic)

| WING AREA           | 608.000    | (FT | **2)  |             |
|---------------------|------------|-----|-------|-------------|
| WING SPAN           | 42.800     | (FT | )     |             |
| MEAN CHORD          | 15.950     | (FT | )     |             |
| VEHICLE WEIGHT      | 45000.000  | (LB | )     |             |
| IXX                 | 28700.000  | (SL | UG-FT | **2)        |
| IYY 1               | 65100.000  | (SL | UG-FT | **2)        |
| IZZ 1               | 87900.000  | (SL | UG-FT | **2)        |
| IXZ                 | -520.000   | (SL | UG-FT | **2)        |
| IXY                 | 0.000      | (SL | UG-FT | **2)        |
| IYZ                 | 0.000      | (SL | UG-FT | **2)        |
| COEFFICIENT OF LIFT |            |     | =     | 0.20709     |
| COEFFICIENT OF DRAG |            |     | =     | 0.01468     |
| LIFT                | (LBS)      |     | =     | 45001.75037 |
| DRAG                | (LBS)      |     | =     | 3190.16917  |
| ALTITUDE            | (FT)       |     | =     | 1000.       |
| MACH                |            |     | =     | 0.50000     |
| VELOCITY            | (FT/SEC)   |     | =     | 556.29559   |
| EQUIVALENT AIRSPEED | (KTS)      |     | =     | 324.60706   |
| SPEED OF SOUND      | (FT/SEC)   |     | =     | 1112.59123  |
| GRAVITATIONAL ACCEL | (FT/SEC**2 | 2)  | =     | 32.17095    |
| NORMAL ACCELERATION | (G-S)      |     | =     | 1.00026     |
| LOAD FACTOR         |            |     | =     | 1.00013     |
| DYNAMIC PRESSURE    | (LBS/FT**2 | 2)  | =     | 357.41711   |
| DENSITY             | (SLUG/FT** | 3)  | =     | 0.00230990  |
| WEIGHT (@ALTITUDE)  | (LBS)      |     | =     | 44995.73786 |
| BETA                | (DEG)      |     | =     | 0.00000     |
| ALPHA               | (DEG)      |     | =     | 0.18105     |
| PHI                 | (DEG)      |     | =     | 0.00000     |
| THETA               | (DEG)      |     | =     | 0.18105     |
| ALTITUDE RATE       | (FT/SEC)   |     | =     | 0.00000     |
| GAMMA               | (DEG)      |     | =     | 0.00000     |
| ROLL RATE           | (DEG/SEC)  |     | =     | 0.00000     |
| PITCH RATE          | (DEG/SEC)  |     | =     | 0.00000     |

| YAW RATE          | (DEG/SEC) | = | 0.00000    |
|-------------------|-----------|---|------------|
| THRUST            | (LBS)     | = | 3188.85962 |
|                   |           |   |            |
| CONTROL VARIABLES |           |   |            |
| ELEVATOR          |           | = | 0.05994    |
| THROTTLE          |           | = | 0.06643    |
| SPEED BRAKE       |           | = | 0.00000    |
| RUDDER            |           | = | 0.00000    |
| AILERON           |           | = | 0.00000    |
| DIFFERENTIAL TAIL |           | = | 0.00000    |

#### NON-DIMENSIONAL STABILITY AND CONTROL DERIVATIVES

|              |        |             |              | SIDE         |
|--------------|--------|-------------|--------------|--------------|
|              |        | DRAG        | LIFT         | FORCE        |
|              |        |             |              |              |
| ZERO COEFFIC | CIENTS | 1.08760D-02 | 1.57360D-01  | 2.12070D-18  |
| ROLL RATE    |        | 0.0000D+00  | 0.0000D+00   | 0.0000D+00   |
| PITCH RATE   |        | 0.0000D+00  | -1.72322D+01 | 0.0000D+00   |
| YAW RATE     |        | 0.0000D+00  | 0.0000D+00   | 0.0000D+00   |
| MACH NUMBER  |        | 0.0000D+00  | 7.45058D-06  | 0.0000D+00   |
| ALPHA        | (RAD)  | 3.72570D-01 | 4.87060D+00  | 0.0000D+00   |
| BETA         | (RAD)  | 0.0000D+00  | 0.0000D+00   | -9.74030D-01 |
| ALTITUDE     | (FT)   | 0.0000D+00  | 0.0000D+00   | 0.0000D+00   |
| ALPHA DOT    |        | 0.0000D+00  | 1.72322D+01  | 0.0000D+00   |
| BETA DOT     |        | 0.0000D+00  | 0.0000D+00   | 0.0000D+00   |
| ELEVATOR     |        | 4.38308D-02 | 5.72957D-01  | 0.00000D+00  |
| SPEED BRAKE  |        | 6.49346D-02 | 3.74913D-02  | 0.0000D+00   |
| RUDDER       |        | 0.0000D+00  | 0.0000D+00   | -1.50410D-01 |
| AILERON      |        | 0.00000D+00 | 0.0000D+00   | -1.15160D-03 |
| DIFFERENTIAL | TAIL   | 0.00000D+00 | 0.0000D+00   | -7.93150D-02 |

NON-DIMENSIONAL STABILITY AND CONTROL DERIVATIVES

|              |       | ROLLING<br>MOMENT | PITCHING<br>MOMENT | YAWING<br>MOMENT |
|--------------|-------|-------------------|--------------------|------------------|
|              |       |                   | <del></del>        |                  |
| ZERO COEFFIC | IENTS | -2.75542D-20      | 4.22040D-02        | -6.80788D-20     |
| ROLL RATE    |       | -2.0000D-01       | 0.0000D+00         | -3.37210D-02     |
| PITCH RATE   |       | 0.0000D+00        | 3.89530D+00        | 0.0000D+00       |
| YAW RATE     |       | 1.50990D-01       | 0.0000D+00         | -4.04710D-01     |
| MACH NUMBER  |       | 2.59775D-13       | -7.05586D-06       | 2.32475D-12      |
| ALPHA        | (RAD) | 0.0000D+00        | -1.68819D-01       | 0.0000D+00       |
| BETA         | (RAD) | -1.33450D-01      | 0.0000D+00         | 1.29960D-01      |
| ALTITUDE     | (FT)  | 0.0000D+00        | 0.0000D+00         | 0.0000D+00       |
| ALPHA DOT    |       | 0.0000D+00        | -1.18870D+01       | 0.0000D+00       |
| BETA DOT     |       | 0.0000D+00        | 0.0000D+00         | 0.0000D+00       |
| ELEVATOR     |       | 0.0000D+00        | -6.95281D-01       | 0.0000D+00       |
| SPEED BRAKE  |       | 0.0000D+00        | -4.17500D-01       | 0.00000D+00      |
| RUDDER       |       | -2.38590D-03      | 0.0000D+00         | -6.97630D-02     |
| AILERON      |       | 2.63560D-02       | 0.0000D+00         | 2.19170D-03      |
| DIFFERENTIAL | TAIL  | 4.01070D-02       | 0.00000D+00        | 3.05310D-02      |

VEHICLE STATIC MARGIN IS 3.5% MEAN AERODYNAMIC CHORD STABLE

## 11.2 Generic F-15 Model Data (Supersonic)

| WING AREA           | 608.000           | (FT**2) |             |
|---------------------|-------------------|---------|-------------|
| WING SPAN           | 42.800            | (FT)    |             |
| MEAN CHORD          | 15.950            | (FT)    |             |
| VEHICLE WEIGHT      | 45000.000         | (LB)    |             |
| IXX                 | 28700.000         | (SLUG-F | T**2)       |
| IYY                 | 165100.000        | (SLUG-F | T**2)       |
| IZZ                 | 187900.000        | (SLUG-F | T**2)       |
| IXZ                 | -520.000          | (SLUG-F | T**2)       |
| IXY                 | 0.000             | (SLUG-F | T**2)       |
| IYZ                 | 0.000             | (SLUG-F | T**2)       |
| COEFFICIENT OF LIFT |                   | =       | 0.05540     |
| COEFFICIENT OF DRAG |                   | =       | 0.00308     |
| LIFT                | (LBS)             | =       | 44992.04685 |
| DRAG                | (LBS)             | =       | 2498.81524  |
| ALTITUDE            | (FT)              | =       | 20000.      |
| MACH                | ,                 | =       | 1.40000     |
| VELOCITY            | (FT/SEC)          | =       | 1451.69421  |
| EQUIVALENT AIRSPEED |                   | =       | 627.54694   |
| SPEED OF SOUND      | (FT/SEC)          | =       | 1036.92440  |
| GRAVITATIONAL ACCEL |                   | 2) =    | 32.11294    |
| NORMAL ACCELERATION | · ·               | = -     | 0.99780     |
| LOAD FACTOR         | (0 0)             | =       | 1.00172     |
| DYNAMIC PRESSURE    | (LBS/FT**2        |         | 1335.83203  |
| DENSITY             | (SLUG/FT**        | -       | 0.00126774  |
| WEIGHT (@ALTITUDE)  | (LBS)             | =       | 44914.60527 |
| BETA                | (DEG)             | =       | 0.00000     |
| ALPHA               | (DEG)             | =       | -1.65589    |
| PHI                 | (DEG)             | =       | 0.00000     |
| THETA               | (DEG)             | =       | -1.65589    |
| ALTITUDE RATE       | (FT/SEC)          | =       | 0.00000     |
| GAMMA               | (PI/SEC)<br>(DEG) | =       | 0.00000     |
| ROLL RATE           | (DEG/SEC)         | =       | 0.00000     |
|                     |                   | =       |             |
| PITCH RATE          | (DEG/SEC)         |         | 0.00000     |
| YAW RATE            | (DEG/SEC)         | =       | 0.00000     |
| THRUST              | (LBS)             | =       | 2499.44214  |
| CONTROL VARIABLES   |                   |         |             |
| ELEVATOR            |                   | =       | 0.06772     |
| THROTTLE            |                   | =       | 0.05207     |
| SPEED BRAKE         |                   | =       | 0.00000     |
| RUDDER              |                   | =       | 0.00000     |
| AILERON             |                   | =       | 0.00000     |
| DIFFERENTIAL TAIL   |                   | =       | 0.00000     |

#### NON-DIMENSIONAL STABILITY AND CONTROL DERIVATIVES

|             |           |             |              | SIDE         |
|-------------|-----------|-------------|--------------|--------------|
|             |           | DRAG        | LIFT         | FORCE        |
| ZERO COEFFI | CIENTS    | 1.08760D-02 | 1.57360D-01  | 2.12070D-18  |
| ROLL RATE   | (RAD/SEC) | 0.0000D+00  | 0.0000D+00   | 0.0000D+00   |
| PITCH RATE  | (RAD/SEC) | 0.0000D+00  | -1.72320D+01 | 0.0000D+00   |
| YAW RATE    | (RAD/SEC) | 0.0000D+00  | 0.0000D+00   | 0.0000D+00   |
| VELOCITY    | (FT/SEC)  | 0.0000D+00  | 0.0000D+00   | 0.0000D+00   |
| MACH NUMBER | _         | 0.0000D+00  | 0.0000D+00   | 0.0000D+00   |
| ALPHA       | (RAD)     | 3.72570D-01 | 4.87060D+00  | 0.0000D+00   |
| BETA        | (RAD)     | 0.0000D+00  | 0.0000D+00   | -9.74030D-01 |
| ALTITUDE    | (FT)      | 0.0000D+00  | 0.0000D+00   | 0.0000D+00   |
| ALPHA DOT   | (RAD/SEC) | 0.0000D+00  | 1.72320D+01  | 0.0000D+00   |
| BETA DOT    | (RAD/SEC) | 0.0000D+00  | 0.0000D+00   | 0.0000D+00   |
| ELEVATOR    |           | 4.38310D-02 | 5.72959D-01  | 0.0000D+00   |
| THROTTLE    |           | 0.0000D+00  | 0.00000D+00  | 0.00000D+00  |
| SPEED BRAKE | 1         | 6.49351D-02 | 3.74913D-02  | 0.00000D+00  |
| RUDDER      |           | 0.0000D+00  | 0.0000D+00   | -1.50410D-01 |
| AILERON     |           | 0.0000D+00  | 0.00000D+00  | -1.15160D-03 |
| DIFFERENTIA | L TAIL    | 0.0000D+00  | 0.0000D+00   | -7.93150D-02 |

NON-DIMENSIONAL STABILITY AND CONTROL DERIVATIVES

|             |           | ROLLING<br>MOMENT | PITCHING<br>MOMENT | YAWING<br>MOMENT |
|-------------|-----------|-------------------|--------------------|------------------|
| ZERO COEFFI | CIENTS    | 2.77192D-19       | 4.22040D-02        | -1.74198D-19     |
| ROLL RATE   | (RAD/SEC) | -2.0000D-01       | 0.0000D+00         | -3.37210D-02     |
| PITCH RATE  | (RAD/SEC) | 0.0000D+00        | 3.89530D+00        | 0.0000D+00       |
| YAW RATE    | (RAD/SEC) | 1.50990D-01       | 0.0000D+00         | -4.04710D-01     |
| VELOCITY    | (FT/SEC)  | 3.42955D-17       | -1.15312D-10       | 3.06986D-16      |
| MACH NUMBER | 3         | 3.55618D-14       | -1.19570D-07       | 3.18322D-13      |
| ALPHA       | (RAD)     | 0.0000D+00        | -1.68819D-01       | 0.0000D+00       |
| BETA        | (RAD)     | -1.33450D-01      | 0.0000D+00         | 1.29960D-01      |
| ALTITUDE    | (FT)      | 0.0000D+00        | 0.0000D+00         | 0.0000D+00       |
| ALPHA DOT   | (RAD/SEC) | 0.0000D+00        | -1.18870D+01       | 0.0000D+00       |
| BETA DOT    | (RAD/SEC) | 0.0000D+00        | 0.0000D+00         | 0.0000D+00       |
| ELEVATOR    |           | 0.0000D+00        | -6.95279D-01       | 0.0000D+00       |
| THROTTLE    |           | 0.0000D+00        | 0.0000D+00         | 0.0000D+00       |
| SPEED BRAKE | 2         | 0.0000D+00        | -4.17500D-01       | 0.0000D+00       |
| RUDDER      |           | -2.38590D-03      | 0.0000D+00         | -6.97630D-02     |
| AILERON     |           | 2.63560D-02       | 0.0000D+00         | 2.19170D-03      |
| DIFFERENTIA | AL TAIL   | 4.01070D-02       | 0.0000D+00         | 3.05310D-02      |
|             |           |                   |                    |                  |

VEHICLE STATIC MARGIN IS 3.5% MEAN AERODYNAMIC CHORD STABLE